CÁLCULO III
CUADERNO DE EJERCICIOS
SOLUCIONES

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INTRODUCCIÓN

Este documento constituye un material de apoyo para el curso de Cálculo III para las carreras de Economía y Dirección Financiera en el ITAM. Contiene las soluciones detalladas del documento de trabajo *Cálculo III, Cuaderno de Ejercicios*, Lorena Zogaib, Departamento de Matemáticas, ITAM, agosto 4 de 2014.

Todas las soluciones fueron elaboradas por mí, sin una revisión cuidadosa, por lo que seguramente el lector encontrará varios errores en el camino. Ésta es una transcripción en computadora, de mis versiones manuscritas originales. Para este fin, tuve la suerte de contar con la colaboración de dos estudiantes de Economía del ITAM: Angélica Martínez Leyva, que realizó la primera transcripción de las soluciones en Scientific WorkPlace, y Rigel Jarabo García, que elaboró la gran mayoría del material gráfico. Quiero expresar mi agradecimiento a ellas, por el entusiasmo y esmero con que llevaron a cabo su trabajo.

Agradezco de antemano sus comentarios y correcciones en relación con este material.

Lorena Zogaib

CÁLCULO III
TAREA 1 - SOLUCIONES
INTEGRAL INDEFINIDA. INTEGRACIÓN POR SUSTITUCIÓN
(Tema 1.1)

1. (a)

$$\begin{aligned}\int \frac{-3}{\sqrt[5]{x^4}}\,dx &= -3\int x^{-4/5}\,dx \\ &= -3\left(\frac{x^{1/5}}{\frac{1}{5}}\right)+C \\ &= -15x^{1/5}+C.\end{aligned}$$

(b)

$$\begin{aligned}\int \sqrt{\sqrt{\sqrt{x}}}\,dx &= \int \left(\left(x^{1/2}\right)^{1/2}\right)^{1/2}\,dx \\ &= \int x^{1/8}\,dx \\ &= \left(\frac{x^{9/8}}{\frac{9}{8}}\right)+C \\ &= \frac{8}{9}x^{9/8}+C.\end{aligned}$$

(c)

$$\begin{aligned}\int \left(\frac{x}{5}-\frac{2}{x^3}+2\right)\,dx &= \frac{1}{5}\int x\,dx-2\int x^{-3}\,dx+2\int\,dx \\ &= \frac{1}{5}\left(\frac{x^2}{2}\right)-2\left(\frac{x^{-2}}{-2}\right)+2x+C \\ &= \frac{x^2}{10}+\frac{1}{x^2}+2x+C.\end{aligned}$$

(d)

$$\begin{aligned}\int \left(\sqrt{x}+\frac{3}{x}\right)^2\,dx &= \int \left(x+\frac{6}{\sqrt{x}}+\frac{9}{x^2}\right)\,dx \\ &= \int x\,dx+6\int x^{-1/2}dx+9\int x^{-2}\,dx \\ &= \frac{x^2}{2}+6\left(\frac{x^{1/2}}{\frac{1}{2}}\right)+\left(\frac{9x^{-1}}{-1}\right)+C \\ &= \frac{x^2}{2}+12\sqrt{x}-\frac{9}{x}+C\end{aligned}$$

(e)

$$\begin{aligned}\int \left(\sqrt{e^{2x}} + \ln 2\right)\ dx &= \int \left(\left(e^{2x}\right)^{1/2} + \ln 2\right)\ dx \\ &= \int \left(e^{x} + \ln 2\right)\ dx \\ &= e^{x} + (\ln 2)\, x + C \\ &= e^{x} + x\ln 2 + C.\end{aligned}$$

2. (a) $f'(x) = 7x + \cos x$.

$$\begin{aligned}f(x) &= \int f'(x)\ dx \\ &= \int (7x + \cos x)\ dx \\ &= \frac{7x^2}{2} + senx + C.\end{aligned}$$

(b) $f''(x) = 9x^2 + 6x$.

$$\begin{aligned}f'(x) &= \int f''(x)\ dx \\ &= \int \left(9x^2 + 6x\right)\ dx \\ &= 3x^3 + 3x^2 + C_1,\end{aligned}$$

$$\begin{aligned}f(x) &= \int f'(x)\, dx \\ &= \int \left(3x^3 + 3x^2 + C_1\right) dx \\ &= \frac{3}{4}x^4 + x^3 + C_1 x + C_2.\end{aligned}$$

(c) $f''(x) = -1$.

$$\begin{aligned}f'(x) &= \int f''(x)\ dx \\ &= \int (-1)\ dx \\ &= -x + C_1,\end{aligned}$$

$$\begin{aligned}f(x) &= \int f'(x)\ dx \\ &= \int (-x + C_1)\ dx \\ &= -\frac{x^2}{2} + C_1 x + C_2.\end{aligned}$$

(d) $f''(x) = 0.$

$$\begin{aligned} f'(x) &= \int f''(x)\ dx \\ &= \int 0\ dx = C_1, \end{aligned}$$

$$\begin{aligned} f(x) &= \int f'(x)\, dx \\ &= \int C_1\ dx \\ &= C_1 x + C_2. \end{aligned}$$

3. (a) $\dfrac{dy}{dx} = \dfrac{1}{x^2} + x,\ \ y(2) = 1.$

Se tiene

$$y(x) = \int \left(\frac{dy}{dx}\right)\ dx = \int \left(\frac{1}{x^2} + x\right)\ dx = -\frac{1}{x} + \frac{x^2}{2} + C.$$

Sabemos que

$$y(2) = 1 = -\frac{1}{2} + \frac{2^2}{2} + C \qquad \therefore \qquad C = -\frac{1}{2}.$$

Por lo tanto,

$$y(x) = -\frac{1}{x} + \frac{x^2}{2} - \frac{1}{2}.$$

(b) $\dfrac{d^2 s}{dt^2} = \dfrac{3t}{8},\ \ s(4) = 4,\ \ s'(4) = 3.$

Se tiene

$$\begin{aligned} s'(t) &= \int s''(t)\ dt = \int \frac{3t}{8}\ dt = \frac{3}{16}t^2 + C_1, \\ s(t) &= \int s'(t)\ dt = \int \left(\frac{3}{16}t^2 + C_1\right)\ dt = \frac{t^3}{16} + C_1 t + C_2. \end{aligned}$$

Sabemos que

$$\begin{aligned} s(4) &= 4 = \frac{4^3}{16} + 4C_1 + C_2 \\ s'(4) &= 3 = \frac{3}{16}4^2 + C_1 \\ &\therefore \quad C_1 = 0, \quad C_2 = 0. \end{aligned}$$

Por lo tanto,

$$s(t) = \frac{t^3}{16}.$$

4. Sabemos que la utilidad marginal es $u'(q) = 100 - 2q$ y que $u(10) = 700$. Para maximizar la utilidad $u(q)$ es necesario que $u'(q) = 0$. Entonces,

$$u'(q) = 0 = 100 - 2q \quad \therefore \quad q = 50.$$

Observamos que se trata de un problema de maximización, ya que

$$u''(q) = -2 < 0.$$

Para encontrar la utilidad máxima $u_{máx}$ debemos encontrar la función $u(q)$. Se tiene

$$u(q) = \int u'(q)\ dq = \int (100 - 2q)\ dq = 100q - q^2 + A.$$

Sabemos que

$$u(10) = 700 = 100(10) - (10)^2 + A \quad \therefore \quad A = -200.$$

Así, la función de utilidad es

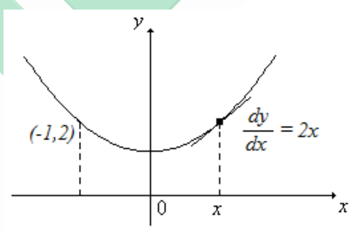
$$u(q) = 100q - q^2 - 200.$$

Concluimos que la empresa maximiza su utilidad en $q = 50$, con

$$u_{máx} = u(50) = 2,300 \text{ dólares}.$$

5. En cualquier punto (x, y) de la curva $y(x)$ la pendiente es el doble de la abscisa, es decir,

$$\frac{dy}{dx} = 2x.$$



De esta manera,

$$y(x) = \int \left(\frac{dy}{dx}\right)\ dx = \int 2x\ dx = x^2 + C.$$

La curva pasa por el punto $(-1, 2)$, de donde

$$y(-1) = 2 = (-1)^2 + C \quad \therefore \quad C = 1.$$

Por lo tanto, la curva que buscamos es

$$y(x) = x^2 + 1.$$

6. (a) $\int \sqrt{1-2x}\, dx.$

Sea
$u = 1-2x$
$du = -2dx.$
Así,

$$\begin{aligned}\int \sqrt{1-2x}\, dx &= -\frac{1}{2}\int \sqrt{1-2x}\ (-2dx) \\ &= -\frac{1}{2}\int \sqrt{u}\, du \\ &= -\frac{1}{3}u^{3/2}+C \\ &= -\frac{1}{3}(1-2x)^{3/2}+C.\end{aligned}$$

(b) $\int \frac{y-1}{\sqrt{y^2-2y+1}}\, dy.$

Sea
$u = y^2-2y+1$
$du = (2y-2)\, dy$
$= 2(y-1)\, dy.$
Así,

$$\begin{aligned}\int \frac{y-1}{\sqrt{y^2-2y+1}}\, dy &= \frac{1}{2}\int \frac{2(y-1)}{\sqrt{y^2-2y+1}}\, dy \\ &= \frac{1}{2}\int \frac{du}{\sqrt{u}} \\ &= \sqrt{u}+C \\ &= \sqrt{y^2-2y+1}+C.\end{aligned}$$

(c) $\int \frac{7r^3}{\sqrt[5]{1-r^4}}\, dr.$

Sea
$u = 1-r^4$
$du = -4r^3dr$
Así,

$$\begin{aligned}\int \frac{7r^3}{\sqrt[5]{1-r^4}}\, dr &= -\frac{7}{4}\int \frac{-4r^3}{\sqrt[5]{1-r^4}}\, dr \\ &= -\frac{7}{4}\int u^{-1/5}\, du \\ &= -\frac{35}{16}u^{4/5}+C \\ &= -\frac{35}{16}\left(1-r^4\right)^{4/5}+C.\end{aligned}$$

(d) $\int \frac{3x}{(3x^2+3)^3}\ dx.$

Sea

$u = 3x^2 + 3$

$du = 6x\ dx$

Así,

$$\begin{aligned}\int \frac{3x}{(3x^2+3)^3}\ dx &= \frac{1}{2}\int \frac{(6x\ dx)}{(3x^2+3)^3} \\ &= \frac{1}{2}\int \frac{du}{u^3} \\ &= \frac{1}{2}\int u^{-3}\ du \\ &= -\frac{1}{4}u^{-2} + C \\ &= -\frac{1}{4(3x^2+3)^2} + C\end{aligned}$$

(e) $\int sen\,(-4x)\ dx.$

Sea

$u = -4x$

$du = -4\ dx.$

Así,

$$\begin{aligned}\int sen\,(-4x)\ dx &= -\frac{1}{4}\int sen\,(-4x)\,(-4\ dx) \\ &= -\frac{1}{4}\int sen\,u\ du \\ &= \frac{1}{4}\cos u + C \\ &= \frac{1}{4}\cos(-4x) + C.\end{aligned}$$

(f) $\int sen^5\,(x/3)\cos(x/3)\ dx.$

Sea

$u = sen\,(x/3)$

$du = \frac{1}{3}\cos(x/3)\ dx.$

Así,

$$\begin{aligned}\int sen^5\,(x/3)\cos(x/3)\ dx &= 3\int sen^5\,(x/3)\left[\frac{1}{3}\cos(x/3)\ dx\right] \\ &= 3\int u^5\ du \\ &= \frac{u^6}{2} + C \\ &= \frac{1}{2}sen^6\,(x/3) + C.\end{aligned}$$

(g) $\int 2x^2 \cos\left(x^3\right)\ dx$.

Sea

$$u = x^3$$
$$du = 3x^2\ dx.$$

Así,

$$\begin{aligned} \int 2x^2 \cos\left(x^3\right)\ dx &= \frac{2}{3}\int \cos\left(x^3\right)\left(3x^2\ dx\right) \\ &= \frac{2}{3}\int \cos u\ du \\ &= \frac{2}{3} sen\, u + C \\ &= \frac{2}{3} sen\left(x^3\right) + C. \end{aligned}$$

(h) $\int \frac{1}{x^2}\cos^2\left(\frac{1}{x}\right)\ dx$.

Sea

$$u = \frac{1}{x}$$
$$du = -\frac{1}{x^2}\ dx.$$

Así,

$$\begin{aligned} \int \frac{1}{x^2}\cos^2\left(\frac{1}{x}\right)\ dx &= -\int \cos^2\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\ dx\right) \\ &= -\int \cos^2 u\ du \\ &= -\left[\frac{u}{2} + \frac{sen\,(2u)}{4}\right] + C \\ &= -\frac{1}{2x} - \frac{1}{4} sen\left(\frac{2}{x}\right) + C. \end{aligned}$$

(i) $\int \cos x \sqrt{1 - 2\, sen\, x}\ dx$.

Sea

$$u = 1 - 2\, sen\, x$$
$$du = -2\cos x\ dx.$$

Así,

$$\begin{aligned} \int \cos x \sqrt{1 - 2\, sen\, x}\ dx &= -\frac{1}{2}\int \sqrt{1 - 2\, sen\, x}(-2\cos x)\ dx \\ &= -\frac{1}{2}\int \sqrt{u}\ du \\ &= -\frac{1}{3}u^{3/2} + C \\ &= -\frac{1}{3}\left(1 - 2\, sen\, x\right)^{3/2} + C. \end{aligned}$$

(j) $\int \frac{1}{\sqrt{x}\,(1+\sqrt{x})^2}\,dx.$

Sea

$$u = 1+\sqrt{x}$$
$$du = \frac{1}{2\sqrt{x}}\,dx.$$

Así,

$$\begin{aligned}
\int \frac{1}{\sqrt{x}\,(1+\sqrt{x})^2}\,dx &= 2\int \frac{\left(\frac{1}{2\sqrt{x}}\,dx\right)}{(1+\sqrt{x})^2} \\
&= 2\int \frac{du}{u^2} \\
&= -\frac{2}{u}+C \\
&= -\frac{2}{1+\sqrt{x}}+C.
\end{aligned}$$

(k) $\int x\sqrt{3x+2}\,dx.$

Sea

$$u = 3x+2$$
$$du = 3\,dx$$
$$x = \frac{u-2}{3}.$$

Así,

$$\begin{aligned}
\int x\sqrt{3x+2}\,dx &= \frac{1}{3}\int x\sqrt{3x+2}\,(3\,dx) \\
&= \frac{1}{3}\int \left(\frac{u-2}{3}\right)\sqrt{u}\,du \\
&= \frac{1}{9}\int \left(u^{3/2}-2u^{1/2}\right)\,du \\
&= \frac{2}{45}u^{5/2}-\frac{4}{27}u^{3/2}+C \\
&= \frac{2}{45}(3x+2)^{5/2}-\frac{4}{27}(3x+2)^{3/2}+C.
\end{aligned}$$

(l) $\int \dfrac{e^{2x}}{e^2+e^{2x}}\,dx.$

Sea

$$u = e^2+e^{2x}$$
$$du = \left(0+2e^{2x}\right)\,dx = 2e^{2x}\,dx$$

Así,

$$\begin{aligned}\int \frac{e^{2x}}{e^2+e^{2x}}\,dx &= \frac{1}{2}\int \frac{2e^{2x}\,dx}{e^2+e^{2x}}\\ &= \frac{1}{2}\int \frac{du}{u}\\ &= \frac{1}{2}\ln|u|+C\\ &= \frac{1}{2}\ln\left|e^2+e^{2x}\right|+C\\ &= \frac{1}{2}\ln(e^2+e^{2x})+C.\end{aligned}$$

(m) $\displaystyle\int \frac{1}{e^x}\sqrt{1+\frac{1}{e^x}}\,dx = \int e^{-x}\sqrt{1+e^{-x}}\,dx.$

Sea

$$u = 1+e^{-x}$$
$$du = -e^{-x}\,dx$$

Así,

$$\begin{aligned}\int \frac{1}{e^x}\sqrt{1+\frac{1}{e^x}}\,dx &= \int e^{-x}\sqrt{1+e^{-x}}\,dx\\ &= -\int \sqrt{1+e^{-x}}\left(-e^{-x}\right)dx\\ &= -\int \sqrt{u}\,du\\ &= -\frac{2}{3}u^{3/2}+C\\ &= -\frac{2}{3}\left(1+e^{-x}\right)^{3/2}+C.\end{aligned}$$

(n) $\displaystyle\int \frac{2^{1/x}}{x^2}\,dx.$

Sea

$$u = \frac{1}{x}$$
$$du = -\frac{1}{x^2}\,dx$$

Así,

$$\begin{aligned}\int \frac{2^{1/x}}{x^2}\,dx &= -\int 2^{1/x}\left(-\frac{1}{x^2}\right)dx\\ &= -\int 2^u\,du\\ &= -\frac{1}{\ln 2}\,2^u + C\\ &= -\frac{1}{\ln 2}\,2^{1/x} + C.\end{aligned}$$

(o) $\displaystyle\int \frac{\ln(4\sqrt{x})}{x}\,dx.$

Sea

$u = \ln(4\sqrt{x}) = \ln 4 + \frac{1}{2}\ln x$

$du = \dfrac{1}{2x}\,dx$

Así,

$$\begin{aligned}\int \frac{\ln(4\sqrt{x})}{x}\,dx &= \int 2\ln(4\sqrt{x})\left(\frac{1}{2x}\right)dx\\ &= \int 2u\,du\\ &= u^2 + C\\ &= \ln^2\left(4\sqrt{x}\right) + C.\end{aligned}$$

(p) $\int (e^x + e^{-x})^2\,dx = \int \left[(e^x)^2 + 2(e^x)(e^{-x}) + (e^{-x})^2\right]dx = \int \left[e^{2x} + 2 + e^{-2x}\right]dx$

$= \int e^{2x}\,dx + \int 2\,dx + \int e^{-2x}\,dx.$

Sean

$u = 2x$ y $w = -2x$

$du = 2\,dx$ $dw = -2\,dx.$

Así,

$$\begin{aligned}\int (e^x + e^{-x})^2\,dx &= \frac{1}{2}\int e^{2x}(2\,dx) + \int 2\,dx - \frac{1}{2}\int e^{-2x}(-2\,dx)\\ &= \frac{1}{2}\int e^u\,du + \int 2\,dx - \frac{1}{2}\int e^w\,dw\\ &= \frac{1}{2}e^u + 2x - \frac{1}{2}e^w + C\\ &= \frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x} + C.\end{aligned}$$

(q) $\int (3^x+1)^2\ dx = \int \left[(3^x)^2 + 2(3^x)(1) + 1\right]\ dx = \int \left[3^{2x} + 2(3^x) + 1\right]\ dx.$

Sea

$u = 2x$

$du = 2\ dx.$

Así,

$$\begin{aligned}\int (3^x+1)^2 dx &= \frac{1}{2}\int 3^{2x}(2\ dx) + 2\int 3^x\ dx + \int\ dx \\ &= \frac{1}{2}\int 3^u\ du + 2\int 3^x\ dx + \int\ dx \\ &= \frac{1}{2}\frac{1}{\ln 3}3^u + 2\frac{1}{\ln 3}3^x + x + C \\ &= \frac{1}{2\ln 3}3^{2x} + \frac{2}{\ln 3}3^x + x + C.\end{aligned}$$

(r) $\int x^{3\alpha}\ d\alpha.$

Sea

$u = 3\alpha$

$du = 3\ d\alpha.$

Así,

$$\begin{aligned}\int x^{3\alpha}\ d\alpha &= \frac{1}{3}\int x^{3\alpha}(3\ d\alpha) \\ &= \frac{1}{3}\int x^u\ du \\ &= \frac{1}{3}\frac{1}{\ln x}x^u + C \\ &= \frac{1}{3}\frac{1}{\ln x}x^{3\alpha} + C.\end{aligned}$$

7. (a) Sean $a \neq 0$, $p \neq -1$. Sea

$u = ax + b$

$du = a\ dx.$

Así,

$$\begin{aligned}\int (ax+b)^p\ dx &= \frac{1}{a}\int (ax+b)^p (a\ dx) \\ &= \frac{1}{a}\int u^p\ du \\ &= \frac{1}{a}\left(\frac{u^{p+1}}{p+1}\right) + C \\ &= \frac{1}{a(p+1)}(ax+b)^{p+1} + C.\end{aligned}$$

(b) i) $\int (2x+1)^4 \ dx$.

Identificamos $a=2$, $b=1$ y $p=4$. Así,

$$\begin{aligned}\int (2x+1)^4 \ dx &= \frac{1}{2(4+1)}(2x+1)^{4+1}+C \\ &= \frac{1}{10}(2x+1)^5+C.\end{aligned}$$

ii) $\int \frac{1}{\sqrt{4-x}} \, dx$.

Identificamos $a=-1$, $b=4$ y $p=-1/2$. Así,

$$\begin{aligned}\int \frac{1}{\sqrt{4-x}} \, dx &= \frac{1}{(-1)(-\frac{1}{2}+1)}(4-x)^{(-1/2)+1}+C \\ &= -2\sqrt{4-x}+C.\end{aligned}$$

CÁLCULO III
TAREA 2 - SOLUCIONES
SUMAS FINITAS. SUMAS DE RIEMANN. INTEGRAL DEFINIDA.
(Tema 1.2)

1. (a) $\displaystyle\sum_{k=1}^{100} 2^{k-1} = 1 + 2 + 2^2 + \cdots + 2^{99}.$

   (b) $\displaystyle\sum_{k=1}^{100} (-1)^{k+1} 2^{k-1} = 1 - 2 + 2^2 + \cdots - 2^{99}.$

   (c) $\displaystyle\sum_{k=1}^{100} (-1)^{k} 2^{k-1} = -1 + 2 - 2^2 + \cdots + 2^{99}.$

   (d) $\displaystyle\sum_{i=0}^{n} (-1)^{i} \cos(ix) = 1 - \cos(x) + \cos(2x) - \cos(3x) + \cdots + (-1)^n \cos(nx).$

   (e) $\displaystyle\sum_{n=j}^{j+2} n^2 = j^2 + (j+1)^2 + (j+2)^2.$

   (f) $\displaystyle\sum_{k=1}^{n} f(x_k)\,\Delta x_k = f(x_1)\,\Delta x_1 + f(x_2)\,\Delta x_2 + \cdots + f(x_n)\,\Delta x_n.$

2. (a) $\displaystyle\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots + \frac{1}{50} = \sum_{k=1}^{25} \frac{1}{2k}.$

   (b) $\displaystyle 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \cdots - \frac{1}{15} = \sum_{k=1}^{8} (-1)^{k+1} \frac{1}{2k-1}.$

   (c) $\displaystyle -\frac{1}{5} + \frac{2}{5} - \frac{3}{5} + \frac{4}{5} - \frac{5}{5} = \frac{1}{5}(-1 + 2 - 3 + 4 - 5) = \frac{1}{5}\sum_{k=1}^{5} (-1)^k\, k = -\frac{1}{5}\sum_{k=1}^{5} (-1)^{k+1}\, k.$

   (d) $\displaystyle x^{-2} + x^{-1} + x^0 + x^1 = \sum_{k=1}^{4} x^{k-3}.$

3. i) $\displaystyle\sqrt{3} - \sqrt{4} + \sqrt{5} - \sqrt{6} = \sum_{k=1}^{4} (-1)^{k+1} \sqrt{k+2}.$

   ii) $\displaystyle\sqrt{3} - \sqrt{4} + \sqrt{5} - \sqrt{6} = \sum_{k=3}^{6} (-1)^{k+1} \sqrt{k}.$

   iii) $\displaystyle\sqrt{3} - \sqrt{4} + \sqrt{5} - \sqrt{6} = \sum_{k=-5}^{-2} (-1)^{k+1} \sqrt{k+8}.$

4. (a) $\sum_{k=4}^{6}(-1)^k\, 3^{6-k} = 3^2 - 3 + 1 = 7.$

(b) $\sum_{k=1}^{3} 3^k - \sum_{k=1}^{3} k^3 = (3 + 3^2 + 3^3) - (1^3 + 2^3 + 3^3) = 3.$

(c) $\sum_{k=1}^{n} 2k(1+3k) = 2\left(\sum_{k=1}^{n} k\right) + 6\left(\sum_{k=1}^{n} k^2\right) = \frac{2n\,(n+1)}{2} + \frac{6n\,(n+1)\,(2n+1)}{6} = 2n\,(n+1)^2\,.$

(d)

$$\begin{aligned}\sum_{k=3}^{99}\left(\frac{1}{k} - \frac{1}{k+1}\right) &= (\frac{1}{3}\underbrace{-\frac{1}{4}) + (\frac{1}{4}}\underbrace{-\frac{1}{5}) + (\frac{1}{5}} - \cdots \underbrace{-\frac{1}{98}) + (\frac{1}{98}}\underbrace{-\frac{1}{99}) + (\frac{1}{99}} - \frac{1}{100})\\ &= \frac{1}{3} - \frac{1}{100} = \frac{97}{300}.\end{aligned}$$

(e)

$$\begin{aligned}\sum_{k=1}^{100}(-1)^{k-1}\left(\frac{1}{2}\right)^{k-1} &= \sum_{k=1}^{100}\left(-\frac{1}{2}\right)^{k-1}\\ &= 1 + \left(-\frac{1}{2}\right) + \left(-\frac{1}{2}\right)^2 + \cdots + \left(-\frac{1}{2}\right)^{99}\\ &= \frac{1 - \left(-\frac{1}{2}\right)^{100}}{1 - \left(-\frac{1}{2}\right)} = \frac{2}{3}\left[1 - \left(\frac{1}{2}\right)^{100}\right].\end{aligned}$$

(f)

$$\begin{aligned}\sum_{i=1}^{2}\sum_{j=1}^{3}(i+j) &= \sum_{i=1}^{2}\left[(i+1) + (i+2) + (i+3)\right]\\ &= \sum_{i=1}^{2}(3i+6)\\ &= [3\,(1) + 6] + [3\,(2) + 6] = 21.\end{aligned}$$

5.

$$\sum_{i=1}^{n} i = \frac{n\,(n+1)}{2} = 78$$

$\therefore\ n\,(n+1) = 156$

$\therefore\ n^2 + n - 156 = 0$

$\therefore\ (n+13)\,(n-12) = 0$

$\therefore\ n = 12 \quad$ (se descarta $n = -13$).

6. Sean $x_1 = 3$, $x_2 = 4$, $x_3 = 1$.

(a) $\mu = \dfrac{1}{3}\sum_{k=1}^{3} x_i = \dfrac{1}{3}(x_1 + x_2 + x_3) = \dfrac{1}{3}(3+4+1) = \dfrac{8}{3}.$

(b) $$\sigma^2 = \frac{1}{3}\sum_{k=1}^{3} x_i^2 - \left(\frac{1}{3}\sum_{k=1}^{3} x_i\right)^2 = \frac{1}{3}(x_1^2 + x_2^2 + x_3^2) - \left[\frac{1}{3}(x_1 + x_2 + x_3)\right]^2$$
$$= \frac{1}{3}(3^2 + 4^2 + 1^2) - \left[\frac{1}{3}(3+4+1)\right]^2 = \frac{14}{9}.$$

7. (a) $\lim\limits_{\|P\|\to 0} \sum_{k=1}^{n} (2c_k^2 - 5c_k)\,\Delta x_k = \int_0^1 (2x^2 - 5x)\,dx.$

(b) $\lim\limits_{\|P\|\to 0} \sum_{k=1}^{n} \sqrt{4 - c_k^2}\,\Delta x_k = \int_0^1 \sqrt{4 - x^2} dx.$

(c) $\lim\limits_{\|P\|\to 0} \sum_{k=1}^{n} \cos(c_k)\,\Delta x_k = \int_{\pi/2}^{\pi} \cos x \ dx.$

8. Hay que identificar $\lim\limits_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta x = \int_a^b f(x)\ dx$, donde $c_k = a + k\Delta x$ y $\Delta x = \dfrac{b-a}{n}$.

(a) $\lim\limits_{n\to\infty} \sum_{k=1}^{n} \underbrace{\left(\frac{3k}{n}\right)^2}_{f(c_k)} \underbrace{\left(\frac{3}{n}\right)}_{\Delta x}.$

Sea $c_k = \dfrac{3k}{n}$. Se tiene
$$c_k = \frac{3k}{n} \quad \Longrightarrow \quad f(c_k) = c_k^2.$$

Como
$$c_k = \frac{3k}{n} = 0 + k\left(\frac{3}{n}\right) = a + k\Delta x,$$

por lo tanto
$$\begin{aligned} a &= 0, \\ \Delta x &= \frac{3}{n} = \frac{b-a}{n}, \\ b &= 3 + a = 3 + 0 = 3. \end{aligned}$$

Así,
$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{3k}{n}\right)^2 \left(\frac{3}{n}\right) = \int_0^3 x^2\ dx.$$

(b) $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\underbrace{\left(2+\frac{4k}{n}\right)^2}_{f(c_k)}\underbrace{\left(\frac{4}{n}\right)}_{\Delta x}.$

Aquí hay dos respuestas directas:

(b.1) Sea $c_k = 2 + \dfrac{4k}{n}$. Se tiene

$$c_k = 2 + \frac{4k}{n} \quad \Longrightarrow \quad f(c_k) = c_k^2.$$

Como

$$c_k = 2 + \frac{4k}{n} = 2 + k\left(\frac{4}{n}\right) = a + k\Delta x,$$

por lo tanto

$$\begin{aligned} a &= 2, \\ \Delta x &= \frac{4}{n} = \frac{b-a}{n}, \\ b &= 4 + a = 4 + 2 = 6. \end{aligned}$$

Así,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(2+\frac{4k}{n}\right)^2\left(\frac{4}{n}\right) = \int_2^6 x^2\,dx.$$

(b.2) Sea $c_k = \dfrac{4k}{n}$. Se tiene

$$c_k = \frac{4k}{n} \quad \Longrightarrow \quad f(c_k) = (2 + c_k)^2.$$

Como

$$c_k = \frac{4k}{n} = 0 + k\left(\frac{4}{n}\right) = a + k\Delta x,$$

por lo tanto

$$\begin{aligned} a &= 0, \\ \Delta x &= \frac{4}{n} = \frac{b-a}{n}, \\ b &= 4 + a = 4 + 0 = 4. \end{aligned}$$

Así,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(2+\frac{4k}{n}\right)^2\left(\frac{4}{n}\right) = \int_0^4 (2+x)^2\,dx.$$

Nota que las dos respuestas son equivalentes. En efecto, usando la sustitución $u = 2 + x$ en la integral de la segunda respuesta, se obtiene

$$\int_0^4 (2+x)^2\,dx = \int_2^6 u^2\,du.$$

(c) $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \underbrace{\left[4-\left(-1+\frac{3k}{n}\right)^2\right]}_{f(c_k)} \underbrace{\left(\frac{3}{n}\right)}_{\Delta x}$.

Sea $c_k = -1 + \frac{3k}{n}$. Se tiene

$$c_k = -1 + \frac{3k}{n} \quad \Longrightarrow \quad f(c_k) = 4 - c_k^2.$$

Como

$$c_k = -1 + \frac{3k}{n} = -1 + k\left(\frac{3}{n}\right) = a + k\Delta x,$$

por lo tanto

$$\begin{aligned} a &= -1, \\ \Delta x &= \frac{3}{n} = \frac{b-a}{n}, \\ b &= 3 + a = 3 + (-1) = 2. \end{aligned}$$

Así,

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left[4-\left(-1+\frac{3k}{n}\right)^2\right]\left(\frac{3}{n}\right) = \int_{-1}^{2} \left(4-x^2\right)\, dx.$$

(d) $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \underbrace{\left[\frac{1}{1+3k/n}\right]}_{f(c_k)} \underbrace{\left(\frac{3}{n}\right)}_{\Delta x}$.

Aquí hay dos respuestas directas:

(d.1) Sea $c_k = \frac{3k}{n}$. Se tiene

$$c_k = \frac{3k}{n} \quad \Longrightarrow \quad f(c_k) = \frac{1}{1+c_k}.$$

Como

$$c_k = \frac{3k}{n} = 0 + k\left(\frac{3}{n}\right) = a + k\Delta x,$$

por lo tanto

$$\begin{aligned} a &= 0, \\ \Delta x &= \frac{3}{n} = \frac{b-a}{n}, \\ b &= 3 + a = 3 + 0 = 3. \end{aligned}$$

Así,

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{3/n}{1+3k/n} = \int_{0}^{3} \frac{1}{1+x}\, dx.$$

(d.2) Sea $c_k = 1 + \dfrac{3k}{n}$. Se tiene

$$c_k = 1 + \frac{3k}{n} \implies f(c_k) = \frac{1}{c_k}.$$

Como

$$c_k = 1 + \frac{3k}{n} = 1 + k\left(\frac{3}{n}\right) = a + k\Delta x,$$

por lo tanto

$$\begin{aligned} a &= 1, \\ \Delta x &= \frac{3}{n} = \frac{b-a}{n}, \\ b &= 3 + a = 3 + 1 = 4. \end{aligned}$$

Así,

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{3/n}{1 + 3k/n} = \int_1^4 \frac{1}{x}\, dx.$$

(e) $$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{1 + (k/n)^2} = \lim_{n\to\infty} \sum_{k=1}^{n} \underbrace{\left[\frac{1}{1 + (k/n)^2}\right]}_{f(c_k)} \underbrace{\left(\frac{1}{n}\right)}_{\Delta x}.$$

Sea $c_k = \dfrac{k}{n}$. Se tiene

$$c_k = \frac{k}{n} \implies f(c_k) = \frac{1}{1 + c_k^2}.$$

Como

$$c_k = \frac{k}{n} = 0 + k\left(\frac{1}{n}\right) = a + k\Delta x,$$

por lo tanto

$$\begin{aligned} a &= 0, \\ \Delta x &= \frac{1}{n} = \frac{b-a}{n}, \\ b &= 1 + a = 1 + 0 = 1. \end{aligned}$$

Así,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{1 + (k/n)^2} = \int_0^1 \frac{1}{1 + x^2}\, dx.$$

(f) $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^4}{n^5} = \lim_{n\to\infty} \sum_{k=1}^{n} \underbrace{\left(\frac{k}{n}\right)^4}_{f(c_k)} \underbrace{\left(\frac{1}{n}\right)}_{\Delta x}$.

Sea $c_k = \frac{k}{n}$. Se tiene

$$c_k = \frac{k}{n} \quad \Longrightarrow \quad f(c_k) = c_k^4.$$

Como

$$c_k = \frac{k}{n} = 0 + k\left(\frac{1}{n}\right) = a + k\Delta x,$$

por lo tanto

$$\begin{aligned} a &= 0, \\ \Delta x &= \frac{1}{n} = \frac{b-a}{n}, \\ b &= 1 + a = 1 + 0 = 1. \end{aligned}$$

Así,

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^4}{n^5} = \int_0^1 x^4 \, dx.$$

(g) $\lim_{n\to\infty} \frac{1}{n}\left[\left(\frac{1}{n}\right)^5 + \left(\frac{2}{n}\right)^5 + \left(\frac{3}{n}\right)^5 + \ldots + \left(\frac{n}{n}\right)^5\right] = \lim_{n\to\infty} \sum_{k=1}^{n} \underbrace{\left(\frac{k}{n}\right)^5}_{f(c_k)} \underbrace{\left(\frac{1}{n}\right)}_{\Delta x}$.

Sea $c_k = \frac{k}{n}$. Se tiene

$$c_k = \frac{k}{n} \quad \Longrightarrow \quad f(c_k) = f(c_k) = c_k^5.$$

Como

$$c_k = \frac{k}{n} = 0 + k\left(\frac{1}{n}\right) = a + k\Delta x,$$

por lo tanto

$$\begin{aligned} a &= 0, \\ \Delta x &= \frac{1}{n} = \frac{b-a}{n}, \\ b &= 1 + a = 1 + 0 = 1. \end{aligned}$$

Así,

$$\lim_{n\to\infty} \frac{1}{n}\left[\left(\frac{1}{n}\right)^5 + \left(\frac{2}{n}\right)^5 + \left(\frac{3}{n}\right)^5 + \ldots + \left(\frac{n}{n}\right)^5\right] = \int_0^1 x^5 \, dx.$$

(h) $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \dfrac{3/n}{\sqrt{2+5k/n}} = \dfrac{3}{5} \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \underbrace{\left[\dfrac{1}{\sqrt{2+5k/n}}\right]}_{f(c_k)} \underbrace{\left(\dfrac{5}{n}\right)}_{\Delta x}.$

Aquí hay dos respuestas directas:

(h.1) Sea $c_k = \dfrac{5k}{n}$. Se tiene

$$c_k = \frac{5k}{n} \quad \Longrightarrow \quad f(c_k) = \frac{1}{\sqrt{2+c_k}}.$$

Como

$$c_k = \frac{5k}{n} = 0 + k\left(\frac{5}{n}\right) = a + k\Delta x,$$

por lo tanto

$$\begin{aligned} a &= 0, \\ \Delta x &= \frac{5}{n} = \frac{b-a}{n}, \\ b &= 5 + a = 5 + 0 = 5. \end{aligned}$$

Así,

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{3/n}{\sqrt{2+5k/n}} = \frac{3}{5} \lim_{n\to\infty} \sum_{k=1}^{n} \frac{5/n}{\sqrt{2+5k/n}} = \frac{3}{5} \int_0^5 \frac{1}{\sqrt{2+x}}\, dx.$$

(h.2) Sea $c_k = 2 + \dfrac{5k}{n}$. Se tiene

$$c_k = 2 + \frac{5k}{n} \quad \Longrightarrow \quad f(c_k) = \frac{1}{\sqrt{c_k}}.$$

Como

$$c_k = 2 + \frac{5k}{n} = 2 + k\left(\frac{5}{n}\right) = a + k\Delta x,$$

por lo tanto

$$\begin{aligned} a &= 2, \\ \Delta x &= \frac{5}{n} = \frac{b-a}{n}, \\ b &= 5 + a = 5 + 2 = 7. \end{aligned}$$

Así,

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{3/n}{\sqrt{2+5k/n}} = \frac{3}{5} \lim_{n\to\infty} \sum_{k=1}^{n} \frac{5/n}{\sqrt{2+5k/n}} = \frac{3}{5} \int_2^7 \frac{1}{\sqrt{x}}\, dx.$$

(i) $$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{3/n}{\left(\sqrt{2+3k/n}\right)(3k/n)}=\lim_{n\to\infty}\sum_{k=1}^{n}\underbrace{\left[\frac{1}{\left(\sqrt{2+3k/n}\right)(3k/n)}\right]}_{f(c_k)}\underbrace{\left(\frac{3}{n}\right)}_{\Delta x}.$$

Sea $c_k=\dfrac{3k}{n}$. Se tiene

$$c_k=\frac{3k}{n}\quad\Longrightarrow\quad f(c_k)=\frac{1}{\left(\sqrt{2+c_k}\right)c_k}.$$

Como

$$c_k=\frac{3k}{n}=0+k\left(\frac{3}{n}\right)=a+k\Delta x,$$

por lo tanto

$$\begin{aligned} a &= 0,\\ \Delta x &= \frac{3}{n}=\frac{b-a}{n},\\ b &= 3+a=3+0=3. \end{aligned}$$

Así,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{3/n}{\left(\sqrt{2+3k/n}\right)(3k/n)}=\int_0^3\frac{1}{\left(\sqrt{2+x}\right)x}\,dx.$$

9. Para calcular la integral $\int_0^1(2x)\ dx$ utilizando sumas de Riemann, observamos que

$$f(c_k)=2c_k,$$

$$a=0,\quad b=1\quad\therefore\quad\Delta x=\frac{b-a}{n}=\frac{1}{n}$$

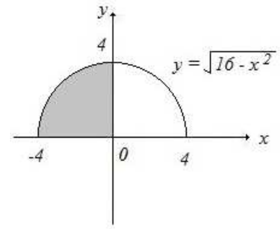
$$\therefore\quad c_k=a+k\Delta x=0+k\left(\frac{1}{n}\right)=\frac{k}{n}.$$

De esta manera,

$$\begin{aligned}\int_0^1(2x)\ dx &= \lim_{\|P\|\to0}\sum_{k=1}^{n}(2c_k)\,\Delta x=\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{2k}{n}\right)\left(\frac{1}{n}\right)=\lim_{n\to\infty}\left(\frac{2}{n^2}\sum_{k=1}^{n}k\right)\\ &= \lim_{n\to\infty}\left(\frac{2}{n^2}\frac{n(n+1)}{2}\right)=\lim_{n\to\infty}\frac{n+1}{n}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)=1.\end{aligned}$$

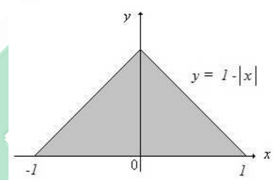
10. (a) La integral $\int_{-4}^{0} \sqrt{16 - x^2}\ dx$ representa la cuarta parte del área de un círculo con centro en el origen y radio igual a 4:

$$\int_{-4}^{0} \sqrt{16 - x^2}\ dx = \frac{1}{4}\left(\pi \cdot 4^2\right) = 4\pi.$$



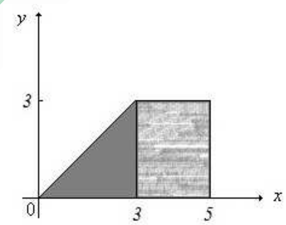
(b) La integral $\int_{-1}^{1} (1 - |x|)\ dx$ representa el área de un triángulo isósceles con base 2 y altura 1:

$$\int_{-1}^{1} (1 - |x|)\ dx = \frac{2\,(1)}{2} = 1.$$



(c) Si $f(x) = \begin{cases} x, & 0 \leq x < 3 \\ 3, & 3 \leq x \leq 5. \end{cases}$, la integral $\int_{0}^{5} f(x)\ dx$ representa el área del trapecio mostrado en la figura:

$$\int_{0}^{5} f(x)\ dx = \frac{3\,(3)}{2} + 2\,(3) = \frac{21}{2}.$$



11. Sea f continua y tal que $\int_{1}^{5} f(x)\ dx = -1$ y $\int_{3}^{5} f(x)\ dx = 4.$

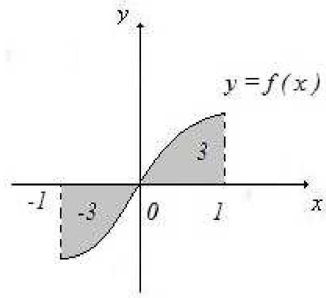
(a) $\int_{1}^{5} f(u)\ du = \int_{1}^{5} f(x)\ dx = -1.$

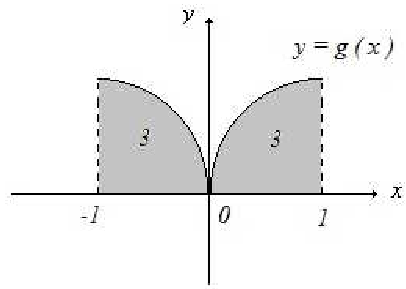
(b) $\int_{1}^{1} f(x)\ dx = \int_{1}^{1} f(x)\ dx = 0.$

(c) $\int_{5}^{1} [-3f(t)]\ dt = -3\int_{5}^{1} f(t)\ dt = 3\int_{1}^{5} f(t)\ dt = 3\,(-1) = -3.$

(d) $\int_{1}^{3} f(x)\ dx = \int_{1}^{5} f(x)\ dx - \int_{3}^{5} f(x)\ dx = (-1) - (4) = -5.$

12. Sea f una función impar y g una par, tales que $\int_{0}^{1} |f(x)| \; dx = \int_{0}^{1} g(x) \; dx = 3.$





(a) $\int_{-1}^{1} f(x) \; dx = \int_{-1}^{0} f(x) \; dx + \int_{0}^{1} f(x) \; dx = -3 + 3 = 0.$

(b) $\int_{-1}^{1} |f(x)| \; dx = \int_{-1}^{0} |f(x)| \; dx + \int_{0}^{1} |f(x)| \; dx = \int_{0}^{1} |f(x)| + \int_{0}^{1} |f(x)| \; dx \; = \; 3 + 3 \; = \; 6.$

(c) $\int_{-1}^{1} g(x) \; dx = \int_{-1}^{0} g(x) \; dx + \int_{0}^{1} g(x) \; dx = 3 + 3 = 6.$

(d) Nota que si $g(x)$ es par, entonces $xg(x)$ es impar (par$\times$par=impar$\times$impar=par; par$\times$impar=impar). Por lo tanto,

$$\int_{-1}^{1} \underbrace{[xg(x)]}_{\text{impar}} \; dx = \int_{-1}^{0} [xg(x)] \; dx + \int_{0}^{1} [xg(x)] \; dx = 0.$$

13. (a) Si f es continua y $f(x) \geq 0$ para todo $x \in [a, b]$, entonces $\int_a^b f(x) \; dx \geq 0$.
Verdadera.

(b) Si $\int_a^b f(x) \; dx \geq 0$, entonces $f(x) \geq 0$ para todo $x \in [a, b]$.
Falsa. Ejemplo: $\int_{-1}^{2} x \; dx = \dfrac{3}{2} > 2$, pero $f(x) = x < 0$ para algunos $x \in [-1, 2]$.

(c) Si $\int_a^b f(x) \; dx$, entonces $f(x) = 0$ para todo $x \in [a, b]$.
Falsa. Ejemplo: $\int_{-1}^{1} x \; dx = 0$, pero $f(x) \neq 0$ para algunos $x \in [-1, 1]$.

(d) Si $f(x) \geq 0$ y $\int_a^b f(x) \; dx = 0$, entonces $f(x) = 0$ para todo $x \in [a, b]$.
Verdadera.

(e) Si $\int_a^b f(x) \; dx > \int_a^b g(x) \; dx$, entonces $\int_a^b [f(x) - g(x)] \; dx > 0$.
Verdadera.

(f) Si f y g son continuas y $f(x) > g(x)$ para todo $x \in [a, b]$, entonces $\left| \int_a^b f(x) \; dx \right| > \left| \int_a^b g(x) \; dx \right|$.
Falsa. Ejemplo: Sean $f(x) = x$ y $g(x) = -2x$. Claramente, $f(x) > g(x)$ en $[1, 2]$. Sin embargo, $\dfrac{3}{2} = \left| \int_1^2 x \; dx \right| < \left| \int_1^2 (-2x) \; dx \right| = 3$.

14. El integrando de $\int_1^3 \frac{1}{1+x^2}\, dx$ es la función continua $f(x) = \frac{1}{1+x^2}$. Los valores mínimo y máximo de f en el intervalo $[1,3]$ son, respectivamente,

$$\min f = \frac{1}{1+3^2} = \frac{1}{10}, \qquad \max f = \frac{1}{1+1} = \frac{1}{2}.$$

Por la desigualdad max-min se tiene

$$\left(\frac{1}{10}\right)(3-1) \leq \int_1^3 \frac{1}{1+x^2}\, dx \leq \left(\frac{1}{2}\right)(3-1),$$

es decir,

$$\frac{1}{5} \leq \int_1^3 \frac{1}{1+x^2}\, dx \leq 1.$$

Por lo tanto, el valor de la integral está entre $1/5$ y 1.

15. El integrando de $\int_0^1 sen\left(x^2\right)\, dx$ es la función continua $f(x) = sen\left(x^2\right)$. Los valores mínimo y máximo de f en el intervalo $[0,1]$ son, respectivamente,

$$\min f = sen(0) = 0, \qquad \max f = sen(1).$$

Por la desigualdad max-min se tiene

$$(0)(1-0) \leq \int_0^1 sen\left(x^2\right)\, dx \leq (sen 1)(1-0),$$

es decir,

$$0 \leq \int_0^1 sen\left(x^2\right)\, dx \leq sen 1.$$

Como

$$sen 1 < 2,$$

por lo tanto

$$\int_0^1 sen\left(x^2\right)\, dx \neq 2.$$

16. (a) $\int_1^3 \pi\, dx = \pi \int_1^3 dx = \pi(3-1) = 2\pi.$

(b) $\int_{-1/2}^{1/2} (-3x)\, dx = -3 \int_{-1/2}^{1/2} x\, dx = -3\left[\frac{(1/2)^2}{2} - \frac{(-1/2)^2}{2}\right] = 0.$

(c)

$$\begin{aligned}\int_{-1}^0 \left[x^2 - \frac{x}{2} + \frac{1}{2}\right] dx &= \int_{-1}^0 x^2\, dx - \frac{1}{2}\int_{-1}^0 x\, dx + \frac{1}{2}\int_{-1}^0 dx \\ &= \left[\frac{0^3}{3} - \frac{(-1)^3}{3}\right] - \frac{1}{2}\left[\frac{0^2}{2} - \frac{(-1)^2}{2}\right] + \frac{1}{2}[0-(-1)] = \frac{13}{12}.\end{aligned}$$

CÁLCULO III
TAREA 3 - SOLUCIONES
TEOREMA FUNDAMENTAL DEL CÁLCULO. SUSTITUCIÓN EN INTEGRAL DEFINIDA
(Temas 1.3-1.4)

1. (a) $G(x) = \int_1^x \sqrt{1+t^4}\, dt.$

$$\frac{dG(x)}{dx} = \frac{d}{dx}\int_1^x \sqrt{1+t^4}\, dt = \sqrt{1+x^4}.$$

(b) $G(x) = \int_x^1 \sqrt{1+t^4}\, dt.$

$$\frac{dG(x)}{dx} = \frac{d}{dx}\int_x^1 \sqrt{1+t^4}\, dt = \frac{d}{dx}\left(-\int_1^x \sqrt{1+t^4}\, dt\right) = -\sqrt{1+x^4}.$$

(c) $G(x) = \int_{-1}^1 \sqrt{1+t^4}\, dt.$

$$\frac{dG(x)}{dx} = \frac{d}{dx}\underbrace{\int_{-1}^1 \sqrt{1+t^4}\, dt}_{\text{cons tan te}} = 0.$$

(d) $G(x) = \int_{-1}^1 x^2\sqrt{1+t^4}\, dt.$

$$\begin{aligned}\frac{dG(x)}{dx} &= \frac{d}{dx}\int_{-1}^1 x^2\sqrt{1+t^4}\, dt = \frac{d}{dx}\,(x^2 \underbrace{\int_{-1}^1 \sqrt{1+t^4}\, dt)}_{\text{cons tan te}} \\ &= \left(\frac{dx^2}{dx}\right)\int_{-1}^1 \sqrt{1+t^4}\, dt = 2x\int_{-1}^1 \sqrt{1+t^4}\, dt.\end{aligned}$$

(e) $G(x) = \int_{-1}^x x^2\sqrt{1+t^4}\, dt.$

$$\begin{aligned}\frac{dG(x)}{dx} &= \frac{d}{dx}\int_{-1}^x x^2\sqrt{1+t^4}\, dt = \frac{d}{dx}\left(x^2\int_{-1}^x \sqrt{1+t^4}\, dt\right) \\ &= x^2\left(\frac{d}{dx}\int_{-1}^x \sqrt{1+t^4}\, dt\right) + \left(\frac{dx^2}{dx}\right)\int_{-1}^x \sqrt{1+t^4}\, dt \\ &= x^2\sqrt{1+x^4} + 2x\int_{-1}^x \sqrt{1+t^4}\, dt.\end{aligned}$$

(f) $G(x) = \int_0^{sen\, x} (u^2 + sen\, u)\ du.$

$$\begin{aligned} \frac{dG(x)}{dx} &= \frac{d}{dx}\int_0^{sen\, x} \left(u^2 + sen\, u\right)\ du = \left[(sen\, x)^2 + sen\,(sen\, x)\right]\frac{d\, sen\, x}{dx} \\ &= \left[sen^2 x + sen\,(sen x)\right]\cos x. \end{aligned}$$

(g) $G(x) = \int_x^{x^3} \sqrt{1+t^4}\ dt.$

Usando la regla de Leibnitz se tiene

$$\begin{aligned} \frac{dG(x)}{dx} &= \frac{d}{dx}\int_x^{x^3} \sqrt{1+t^4}\ dt = \sqrt{1+(x^3)^4}\,\frac{d\,x^3}{dx} - \sqrt{1+x^4}\,\frac{d\,x}{dx} \\ &= \sqrt{1+x^{12}}\left(3x^2\right) - \sqrt{1+x^4}. \end{aligned}$$

(h) $G(x) = \int_0^{x} (x^2+t^3)^9\ dt.$

Usando el caso general de la regla de Leibnitz se tiene

$$\begin{aligned} \frac{dG(x)}{dx} &= \frac{d}{dx}\int_0^{x} \left(x^2+t^3\right)^9\ dt \\ &= \left(x^2+x^3\right)^9\frac{d\,x}{dx} - \left(x^2+0^3\right)^9\frac{d\,0}{dx} + \int_0^x \frac{\partial\,(x^2+t^3)^9}{\partial x}\ dt \\ &= \left(x^2+x^3\right)^9 - 0 + \int_0^x 9\left(x^2+t^3\right)^8(2x)\ dt \\ &= \left(x^2+x^3\right)^9 + 18x\int_0^x \left(x^2+t^3\right)^8\ dt. \end{aligned}$$

(i) $G(x) = \int_{\sqrt{x}}^{x^2} \cos(t^2-x^4)\ dt.$

Usando el caso general de la regla de Leibnitz se tiene

$$\begin{aligned} \frac{dG(x)}{dx} &= \frac{d}{dx}\int_{\sqrt{x}}^{x^2} \cos(t^2-x^4)\ dt \\ &= \cos\left(\left(x^2\right)^2 - x^4\right)\frac{d\,x^2}{dx} - \cos\left(\left(\sqrt{x}\right)^2 - x^4\right)\frac{d\,\sqrt{x}}{dx} + \int_{\sqrt{x}}^{x^2}\frac{\partial\cos(t^2-x^4)}{\partial x}\ dt \\ &= 2x\cos(0) - \frac{1}{2\sqrt{x}}\cos(x-x^4) - \int_{\sqrt{x}}^{x^2} sen(t^2-x^4)\left(-4x^3\right)\ dt \\ &= 2x - \frac{1}{2\sqrt{x}}\cos(x-x^4) + 4x^3\int_{\sqrt{x}}^{x^2} sen(t^2-x^4)\ dt. \end{aligned}$$

2. (a) $r(\theta) = \int\limits_{e^{\sqrt{\theta}}}^{e^{3\theta}} \ln t \, dt.$

$$\frac{dr(\theta)}{d\theta} = \frac{d}{d\theta} \int\limits_{e^{\sqrt{\theta}}}^{e^{3\theta}} \ln t \, dt = \ln\left(e^{3\theta}\right) \frac{de^{3\theta}}{d\theta} - \ln\left(e^{\sqrt{\theta}}\right) \frac{de^{\sqrt{\theta}}}{d\theta} = 3\theta\left(3e^{3\theta}\right) - \sqrt{\theta}\left(\frac{1}{2\sqrt{\theta}} e^{\sqrt{\theta}}\right)$$
$$= 9\theta e^{3\theta} - \frac{1}{2} e^{\sqrt{\theta}}.$$

(b) $\alpha(u) = \int\limits_{0}^{\ln u} sen\left(e^{x}\right) \, dx.$

$$\frac{d\alpha(u)}{du} = \frac{d}{du} \int\limits_{0}^{\ln u} sen\left(e^{x}\right) \, dx = sen\left(e^{\ln u}\right) \frac{d \ln u}{du} = (sen\, u)\left(\frac{1}{u}\right) = \frac{sen\, u}{u}.$$

(c) $G(x) = \int\limits_{0}^{2\ln x} \left(e^{t} + t^{2}\right) \, dt, \quad x > 1.$

$$\frac{dG(x)}{dx} = \frac{d}{dx} \int\limits_{0}^{2\ln x} \left(e^{t} + t^{2}\right) \, dt = \left[e^{2\ln x} + (2\ln x)^{2}\right] \frac{d\, 2\ln x}{dx}$$
$$= \left[e^{\ln x^{2}} + 4(\ln x)^{2}\right] \frac{2}{x} = \frac{2}{x}\left[x^{2} + 4\ln^{2} x\right].$$

(d) $x(y) = \int\limits_{0}^{y} y^{2}\left(\frac{1}{1+s^{3}}\right) \, ds.$

$$\frac{dx(y)}{dy} = \frac{d}{dy} \int\limits_{0}^{y} y^{2}\left(\frac{1}{1+s^{3}}\right) ds = \frac{d}{dy}\left(y^{2} \int\limits_{0}^{y} \frac{1}{1+s^{3}} \, ds\right)$$
$$= \left(\frac{dy^{2}}{dy}\right) \int\limits_{0}^{y} \frac{1}{1+s^{3}} \, ds + y^{2}\left(\frac{d}{dy} \int\limits_{0}^{y} \frac{1}{1+s^{3}} \, ds\right)$$
$$= 2y \int\limits_{0}^{y} \frac{1}{1+s^{3}} \, ds + \frac{y^{2}}{1+y^{3}}.$$

(e) $x(t) = \int\limits_{0}^{t} e^{\left(t^{2}-z^{2}\right)} \, dz.$

$$\frac{dx(t)}{dt} = \frac{d}{dt} \int\limits_{0}^{t} e^{\left(t^{2}-z^{2}\right)} \, dz = \frac{d}{dt}\left(e^{t^{2}} \int\limits_{0}^{t} e^{-z^{2}} \, dz\right)$$
$$= \left(\frac{de^{t^{2}}}{dt}\right) \int\limits_{0}^{t} e^{-z^{2}} \, dz + e^{t^{2}}\left(\frac{d}{dt} \int\limits_{0}^{t} e^{-z^{2}} \, dz\right)$$
$$= \left(2te^{t^{2}}\right) \int\limits_{0}^{t} e^{-z^{2}} \, dz + e^{t^{2}} e^{-t^{2}} = 2t \int\limits_{0}^{t} e^{\left(t^{2}-z^{2}\right)} \, dz + 1.$$

Nota que la respuesta también se puede escribir como

$$\frac{dx(t)}{dt} = 2t\, x(t) + 1.$$

3. Sea $f(x) = \int_0^{1/x} \frac{1}{t^2+1}\, dt + \int_0^{x} \frac{1}{t^2+1}\, dt$, con $x > 0$. Como

$$\begin{aligned} \frac{df(x)}{dx} &= \frac{d}{dx}\int_0^{1/x} \frac{1}{t^2+1}\, dt + \frac{d}{dx}\int_0^{x} \frac{1}{t^2+1}\, dt = \frac{1}{(1/x)^2+1}\cdot \frac{d}{dx}\left(\frac{1}{x}\right) + \frac{1}{x^2+1} \\ &= \frac{x^2}{1+x^2}\left(-\frac{1}{x^2}\right) + \frac{1}{x^2+1} = 0, \end{aligned}$$

por lo tanto f es constante para todo $x > 0$.

4. Sea $G(x) = \int_a^x f(t)\, dt$, con a una constante.

   (a) $\dfrac{dG(x)}{dx} = \dfrac{d}{dx}\int_a^x f(t)\, dt = f(x)$.

   (b) $G(x^2) = \int_a^{x^2} f(t)\, dt$.

   (c) $\dfrac{dG(x^2)}{dx} = \dfrac{d}{dx}\int_a^{x^2} f(t)\, dt = f(x^2)\left(\dfrac{dx^2}{dx}\right) = 2x\, f(x^2)$.

5. Sea $F(x) = \int_4^{x^2} e^{\sqrt{t}}\, dt$, para todo $x \geq 2$.

   (a) $F(2) = \int_4^{4} e^{\sqrt{t}}\, dt = 0$.

   (b) $F'(x) = \dfrac{d}{dx}\int_4^{x^2} e^{\sqrt{t}}\, dt = e^{\sqrt{x^2}}\left(\dfrac{dx^2}{dx}\right) = 2xe^{|x|} = 2xe^{x}$. Nota que $\sqrt{x^2} = |x| = x$, para $x \geq 2$.

   (c) $F''(x) = \dfrac{d}{dx}(2xe^x) = 2xe^x + 2e^x = 2(x+1)e^x$.

6. Sea $f(x) = \int_1^{u(x)} \cos^2 t\, dt$, con $u(x) = \ln(x^2+x-1)$.

$$\begin{aligned} f'(x) &= \frac{d}{dx}\int_1^{u(x)} \cos^2 t\, dt = \cos^2 u(x)\, \frac{du(x)}{dx} = \cos^2\left(\ln\left(x^2+x-1\right)\right)\frac{d\ln(x^2+x-1)}{dx} \\ &= \cos^2\left(\ln\left(x^2+x-1\right)\right)\left(\frac{2x+1}{x^2+x-1}\right), \\ f'(1) &= \cos^2(\ln(1+1-1))\left(\frac{2(1)+1}{1+1-1}\right) = 3\cos^2(\ln 1) = 3\cos^2(0) = 3. \end{aligned}$$

7. Sea $F(x)=\int_1^x f(t)\ dt$, con $f(t)=\int_1^{t^2}\frac{\sqrt{1+u^4}}{u}\ du$.

$$F'(x) = \frac{d}{dx}\int_1^x f(t)\ dt = f(x) = \int_1^{x^2}\frac{\sqrt{1+u^4}}{u}\ du,$$

$$F''(x) = \frac{d}{dx}\int_1^{x^2}\frac{\sqrt{1+u^4}}{u}\ du = \frac{\sqrt{1+(x^2)^4}}{(x^2)}\left(\frac{dx^2}{dx}\right) = \frac{2\sqrt{1+x^8}}{x}.$$

8. Sea $G(x)=\int_2^x\left(s\int_2^s f(t)\ dt\right)ds$, donde f es continua para todo real t.

Definimos $\alpha(s)=s\int_2^s f(t)\ dt$, de modo que $G(x)=\int_2^x \alpha(s)\ ds$.

(a) $G(2)=\int_2^2\alpha(s)\ ds=0$.

(b) $G'(x)=\frac{d}{dx}\int_2^x\alpha(s)\ ds=\alpha(x)=x\int_2^x f(t)\ dt$.

(c) $G'(2)=2\int_2^2 f(t)\ dt=0$.

(d) $G''(x)=\frac{d}{dx}\left(x\int_2^x f(t)\ dt\right)=\left(\frac{dx}{dx}\right)\int_2^x f(t)\ dt+x\left(\frac{d}{dx}\int_2^x f(t)\ dt\right)=\int_2^x f(t)\ dt+x\,f(x)$.

(e) $G''(2)=\int_2^2 f(t)\ dt+2\ f(2)=0+2f(2)=2f(2)$.

9. Sea $F(x)=\int_1^x\left(t-2t\ln\sqrt{t}\right)\ dt,\ \ x\geq 1$. Por lo tanto,

$F'(x)=\frac{d}{dx}\int_1^x\left(t-2t\ln\sqrt{t}\right)\ dt=x-2x\ln\sqrt{x}=x\left(1-2\ln\sqrt{x}\right)=x\left(1-\ln x\right).$

Puntos críticos:

$F'(x)=0\quad\Longrightarrow\quad \underbrace{x=0}_{\text{se descarta ya que }x\geq 1}\quad \text{o}\quad (1-\ln x)=0$

$\therefore\quad \ln x=1$

$\therefore\quad x=e.$

$F''(x)=\frac{d}{dx}\left[x\left(1-\ln x\right)\right]=x\left(-\frac{1}{x}\right)+\left(1-\ln x\right)=-\ln x,$

$F''(e)=-\ln e=-1.$

Como $F''(e)<0$, por lo tanto F es cóncava en $x=e$.

Por lo tanto, F tiene un máximo local en $x=e$.

10. Sea f tal que $f' > 0$ y $f(1) = 0$. Sea $g(x) = \int\limits_0^x f(t)\ dt$.

   (a) g es una funcion diferenciable de x.
   Verdadero.
   Como f es diferenciable, por lo tanto f es continua. Por el Teorema Fundamental de Cálculo, $g(x)$ es diferenciable (con $\dfrac{dg}{dx} = f$).

   (b) g es una funcion continua de x.
   Verdadero.
   Como g es diferenciable, por lo tanto g es continua.

   (c) La gráfica de g tiene una tangente horizontal en $x = 1$.
   Verdadero.
   $\dfrac{dg(x)}{dx} = f(x) \quad \therefore \quad \left.\dfrac{dg(x)}{dx}\right|_{x=1} = f(1) = 0 \quad \therefore$ en $x = 1$ la tangente de g es horizontal.

   (d) g tiene un máximo local en $x = 1$.
   Falso.
   $g''(x) = f'(x) > 0 \ \therefore\ g$ es convexa $\forall x \ \therefore\ g$ tiene un mínimo local en $x = 1$.

   (e) La gráfica de $\dfrac{dg}{dx}$ cruza el eje x en $x = 1$.
   Verdadero.
   $g'(1) = \left.\dfrac{dg(x)}{dx}\right|_{x=1} = f(1) = 0.$

11. Reescribimos el precio $P(t)$, de la siguiente manera:

$$P(t) = \int\limits_t^{20} v(s)\, e^{r(t-s)}\, ds = e^{rt} \int\limits_t^{20} v(s)\, e^{-rs}\, ds = -e^{rt} \int\limits_{20}^{t} v(s)\, e^{-rs}\, ds.$$

En ese caso,

$$\begin{aligned}
\frac{dP(t)}{dt} &= -\left(\frac{de^{rt}}{dt}\right) \int\limits_{20}^{t} v(s)\, e^{-rs}\, ds - e^{rt} \frac{d}{dt} \int\limits_{20}^{t} v(s)\, e^{-rs}\, ds \\
&= -re^{rt} \int\limits_{20}^{t} v(s)\, e^{-rs}\, ds - e^{rt}\left(v(t)\, e^{-rt}\right) \\
&= -r \int\limits_{20}^{t} v(s)\, e^{r(t-s)}\, ds - v(t) \\
&= r \int\limits_{t}^{20} v(s)\, e^{r(t-s)}\, ds - v(t) \\
&= rP(t) - v(t).
\end{aligned}$$

12. De acuerdo con el Teorema Fundamental del Cálculo,

$$\frac{dF(x)}{dx} = \sqrt{5 + e^{x^2}} \quad \Longrightarrow \quad F(x) = \int\limits_a^x \sqrt{5 + e^{t^2}}\, dt,$$

con a una constante. En particular,

$$F(2) = \int_a^2 \sqrt{5 + e^{t^2}}\ dt,$$

de modo que

$$\begin{aligned} F(x) - F(2) &= \int_a^x \sqrt{5 + e^{t^2}}\ dt - \int_a^2 \sqrt{5 + e^{t^2}}\ dt \\ &= \int_a^x \sqrt{5 + e^{t^2}}\ dt + \int_2^a \sqrt{5 + e^{t^2}}\ dt \\ &= \int_2^a \sqrt{5 + e^{t^2}}\ dt + \int_a^x \sqrt{5 + e^{t^2}}\ dt = \int_2^x \sqrt{5 + e^{t^2}}\ dt. \end{aligned}$$

De esta manera,

$$F(x) = \int_2^x \sqrt{5 + e^{t^2}}\ dt + F(2).$$

Como $F(2) = 7$, por lo tanto

$$F(x) = \int_2^x \sqrt{5 + e^{t^2}}\ dt + 7.$$

En efecto,

$$\begin{aligned} F(2) &= \int_2^2 \sqrt{5 + e^{t^2}}\ dt + 7 = 0 + 7 = 7, \\ \frac{dF(x)}{dx} &= \frac{d}{dx}\left(\int_2^x \sqrt{5 + e^{t^2}}\ dt + 7\right) = \sqrt{5 + e^{x^2}}. \end{aligned}$$

13. De acuerdo con el Teorema Fundamental del Cálculo,

$$\frac{dx(t)}{dt} = t^2 f(t) \quad \Longrightarrow \quad x(t) = \int_a^t u^2 f(u)\ du,$$

con a una constante. En particular,

$$x(3) = \int_a^3 u^2 f(u)\ du,$$

de modo que

$$x(t) - x(3) = \int_a^t u^2 f(u)\ du - \int_a^3 u^2 f(u)\ du = \int_3^t u^2 f(u)\ du.$$

De esta manera,

$$x(t) = \int_3^t u^2 f(u)\ du + x(3).$$

Por último, como $x(3) = 5$, por lo tanto

$$x(t) = \int_3^t u^2 f(u)\ du + 5.$$

14. De acuerdo con el Teorema Fundamental del Cálculo,

$$f(t) = \frac{dF(t)}{dt} \quad \Longrightarrow \quad F(t) = \int_a^t f(u)\ du,$$

con a una constante. En particular,

$$F(t_0) = \int_a^{t_0} f(u)\ du,$$

de modo que

$$F(t) - F(t_0) = \int_a^t f(u)\ du - \int_a^{t_0} f(u)\ du = \int_{t_0}^t f(u)\ du.$$

De esta manera,

$$F(t) = \int_{t_0}^t f(u)\ du + F(t_0).$$

Por último, como $F(t_0) = K$, por lo tanto,

$$F(t) = \int_{t_0}^t f(u)\ du + K.$$

15. (a)

$$\begin{aligned}\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) dx &= \int_1^4 \left(\frac{3}{2}x^{1/2} - 4x^{-2}\right) dx = \left[x^{3/2} + \frac{4}{x}\right]_1^4 \\ &= \left(4^{3/2} + \frac{4}{4}\right) - \left(1^{3/2} + \frac{4}{1}\right) = 9 - 5 = 4.\end{aligned}$$

(b)

$$\begin{aligned}\int_{-8}^{-1} \left(5x^{2/3} - 5 - \frac{8}{x^2}\right) dx &= 5\left[\frac{x^{5/3}}{\frac{5}{3}}\right]_{-8}^{-1} - 5\left[x\right]_{-8}^{-1} + 8\left[\frac{1}{x}\right]_{-8}^{-1} \\ &= 3\left[(-1)^{5/3} - (-8)^{5/3}\right] - 5\left[(-1) - (-8)\right] + 8\left[\frac{1}{(-1)} - \frac{1}{(-8)}\right] \\ &= 3\left[(-1) - (-32)\right] - 5\left[7\right] + 8\left[-1 + \frac{1}{8}\right] = 51.\end{aligned}$$

(c)

$$\begin{aligned}\int_{-2}^{-1} \frac{y^5 + 1}{y^3}\, dy &= \int_{-2}^{-1} \left(y^2 + y^{-3}\right)\ dy = \left[\frac{y^3}{3} - \frac{1}{2y^2}\right]_{-2}^{-1} \\ &= \left(-\frac{1}{3} - \frac{1}{2}\right) - \left(-\frac{8}{3} - \frac{1}{8}\right) = \frac{7}{3} - \frac{3}{8} = \frac{47}{24}.\end{aligned}$$

(d)

$$\int_0^1 \left(x^2+2x\right)^2\ dx = \int_0^1 \left(x^4+4x^3+4x^2\right)\ dx = \left[\frac{x^5}{5}+x^4+\frac{4x^3}{3}\right]_0^1 = \frac{38}{15}.$$

(e)

$$\begin{aligned}\int_{-\ln 3}^{\ln 3} (e^x+1)\ dx &= [e^x+x]_{-\ln 3}^{\ln 3} = \left(e^{\ln 3}+\ln 3\right) - \left(e^{-\ln 3}+(-\ln 3)\right) \\ &= \left(e^{\ln 3}+\ln 3\right) - \left(\frac{1}{e^{\ln 3}}-\ln 3\right) = (3+\ln 3) - \left(\frac{1}{3}-\ln 3\right) \\ &= \frac{8}{3}+2\ln 3 = \frac{8}{3}+\ln 9.\end{aligned}$$

(f) Como

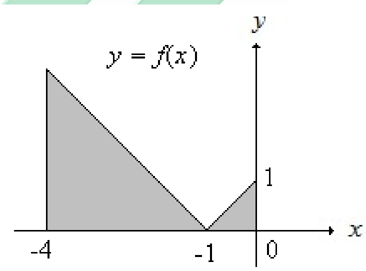
$$|x+1| = \begin{cases} x+1, & x \geq -1, \\ -x-1, & x < -1, \end{cases}$$

por lo tanto,

$$\therefore \quad \int_{-4}^{0} |x+1|\ dx = \int_{-4}^{-1} (-x-1)\ dx + \int_{-1}^{0} (x+1)\ dx = \left[-\frac{x^2}{2}-x\right]_{-4}^{-1} + \left[\frac{x^2}{2}+x\right]_{-1}^{0} = 5.$$

O bien, por sustitución,

$$\int_{-4}^{0} |x+1|\ dx = \int_{-3}^{1} |u|\ du = -\int_{-3}^{0} u\ du + \int_{0}^{1} u\ du = -\left[\frac{u^2}{2}\right]_{-3}^{0} + \left[\frac{u^2}{2}\right]_{0}^{1} = 5.$$



16. (a) $\int_1^3 \sqrt{6-2x}\ dx.$

Sea $u = 6-2x$. Se tiene

$$u = 6-2x \quad \Longrightarrow \quad du = -2\ dx,\ u(1) = 4,\ u(3) = 0.$$

Por lo tanto,

$$\begin{aligned}\int_1^3 \sqrt{6-2x}\ dx &= -\frac{1}{2}\int_1^3 \sqrt{6-2x}\ (-2\ dx) = -\frac{1}{2}\int_4^0 \sqrt{u}\ du = \frac{1}{2}\int_0^4 \sqrt{u}\ du \\ &= \frac{1}{3}\left[u^{3/2}\right]_0^4 = \frac{1}{3}\left[4^{3/2}-0\right] = \frac{8}{3}.\end{aligned}$$

(b) $\displaystyle\int_{\sqrt{2}}^{\sqrt{7}} \frac{x}{\sqrt{x^2+2}}\, dx.$

Sea $u = x^2 + 2$. Se tiene

$$u = x^2 + 2 \quad\Longrightarrow\quad du = 2x\, dx,\ u(\sqrt{2}) = 4,\ u(\sqrt{7}) = 9.$$

Por lo tanto,

$$\int_{\sqrt{2}}^{\sqrt{7}} \frac{x}{\sqrt{x^2+2}}\, dx = \int_{\sqrt{2}}^{\sqrt{7}} \frac{2x\, dx}{2\sqrt{x^2+2}} = \int_4^9 \frac{du}{2\sqrt{u}} = \left[\sqrt{u}\right]_4^9 = 3 - 2 = 1.$$

(c) $\displaystyle\int_1^x \frac{t^2}{\sqrt{t^3+3}}\, dt.$

Sea $u = t^3 + 3$. Se tiene

$$u = t^3 + 3 \quad\Longrightarrow\quad du = 3t^2\, dt,\ u(1) = 4,\ u(x) = x^3 + 3.$$

Por lo tanto,

$$\begin{aligned}\int_1^x \frac{t^2}{\sqrt{t^3+3}}\, dt &= \frac{1}{3}\int_1^x \frac{1}{\sqrt{t^3+3}}\, 3t^2\, dt = \frac{1}{3}\int_{4}^{x^3+3} \frac{du}{\sqrt{u}} \\ &= \frac{2}{3}\left[\sqrt{u}\right]_4^{x^3+3} = \frac{2}{3}\left[\sqrt{x^3+3} - 2\right].\end{aligned}$$

(d) $\displaystyle\int_1^3 \frac{x^2+1}{x^3+3x}\, dx.$

Sea $u = x^3 + 3x$. Se tiene

$$u = x^3 + 3x \quad\Longrightarrow\quad du = \left(3x^2 + 3\right)\, dx,\ u(1) = 4,\ u(3) = 36.$$

Por lo tanto,

$$\begin{aligned}\int_1^3 \frac{x^2+1}{x^3+3x}\, dx &= \frac{1}{3}\int_1^3 \frac{1}{x^3+3x}\left(3x^2+3\right)\, dx = \frac{1}{3}\int_4^{36} \frac{du}{u} \\ &= \frac{1}{3}\left[\ln|u|\right]_4^{36} = \frac{1}{3}\left(\ln 36 - \ln 4\right) = \frac{1}{3}\ln\left(\frac{36}{4}\right) = \frac{1}{3}\ln 9.\end{aligned}$$

(e) $\displaystyle\int_1^4 \frac{dx}{2\sqrt{x}\left(1+\sqrt{x}\right)^2}.$

Sea $u = 1 + \sqrt{x}$. Se tiene

$$u = 1 + \sqrt{x} \quad\Longrightarrow\quad du = \frac{1}{2\sqrt{x}}\, dx,\ u(1) = 2,\ u(4) = 3.$$

Por lo tanto,

$$\begin{aligned}\int_1^4 \frac{dx}{2\sqrt{x}\left(1+\sqrt{x}\right)^2} &= \int_1^4 \frac{1}{\left(1+\sqrt{x}\right)^2}\left(\frac{1}{2\sqrt{x}}\right) dx = \int_2^3 \frac{du}{u^2} = \left[-\frac{1}{u}\right]_2^3 \\ &= -\frac{1}{3} + \frac{1}{2} = \frac{1}{6}.\end{aligned}$$

(f) $\int\limits_{\pi/2}^{\pi} sen\,(2\theta)\;d\theta.$

Sea $u = 2\theta$. Se tiene

$$u = 2\theta \quad \Longrightarrow \quad du = 2\;d\theta,\; u(\frac{\pi}{2}) = \pi,\; u(\pi) = 2\pi.$$

Por lo tanto,

$$\begin{aligned}\int\limits_{\pi/2}^{\pi} sen\,(2\theta)\;d\theta &= \frac{1}{2}\int\limits_{\pi/2}^{\pi} sen\,(2\theta)\;(2\;d\theta) = \frac{1}{2}\int\limits_{\pi}^{2\pi} sen\,u\,du \\ &= \frac{1}{2}\left[-\cos u\right]_{\pi}^{2\pi} = \frac{1}{2}(-\cos(2\pi) + \cos(\pi)) = \frac{1}{2}(-1-1) = -1.\end{aligned}$$

(g) $\int\limits_{-4\ln 3}^{0} \sqrt{e^x}\;dx = \int\limits_{-4\ln 3}^{0} e^{x/2}\;dx.$

Sea $u = \frac{x}{2}$. Se tiene

$$u = \frac{x}{2} \quad \Longrightarrow \quad du = \frac{1}{2}\;dx,\; u(-4\ln 3) = -2\ln 3,\; u(0) = 0.$$

Por lo tanto,

$$\begin{aligned}\int\limits_{-4\ln 3}^{0} \sqrt{e^x}\;dx &= 2\int\limits_{-4\ln 3}^{0} e^{x/2}\left(\frac{1}{2}\;dx\right) = 2\int\limits_{-2\ln 3}^{0} e^u\;du = 2\;\left[e^u\right]_{-2\ln 3}^{0} \\ &= 2(e^0 - e^{-2\ln 3}) = 2(1 - e^{\ln 3^{-2}}) = 2(1 - \frac{1}{9}) = \frac{16}{9}.\end{aligned}$$

(h) $\int\limits_{0}^{\ln 2} \frac{e^{3x}\;dx}{2e^{3x}-1}.$

Sea $u = 2e^{3x} - 1$. Se tiene

$$u = 2e^{3x}-1 \quad \Longrightarrow \quad du = 6e^{3x}\;dx,\; u(0) = 1,\; u(\ln 2) = 2e^{3\ln 2}-1 = 2e^{\ln 8}-1 = 15.$$

Por lo tanto,

$$\begin{aligned}\int\limits_{0}^{\ln 2} \frac{e^{3x}\;dx}{2e^{3x}-1} &= \frac{1}{6}\int\limits_{0}^{\ln 2} \frac{6e^{3x}\;dx}{2e^{3x}-1} = \frac{1}{6}\int\limits_{1}^{15} \frac{du}{u} = \frac{1}{6}\left[\ln|u|\right]_1^{15} = \frac{1}{6}(\ln 15 - \ln 1) \\ &= \frac{1}{6}\ln 15.\end{aligned}$$

(i) $\int\limits_{1}^{e} \frac{dx}{x\,(\ln x + 1)}.$

Sea $u = \ln x + 1$. Se tiene

$$u = \ln x + 1 \quad \Longrightarrow \quad du = \frac{1}{x}\;dx,\; u(1) = \ln 1 + 1 = 1,\; u(e) = \ln e + 1 = 2.$$

Por lo tanto,

$$\int\limits_{1}^{e} \frac{dx}{x\,(\ln x + 1)} = \int\limits_{1}^{e} \frac{1}{\ln x + 1}\left(\frac{1}{x}\right)dx = \int\limits_{1}^{2} \frac{du}{u} = \left[\ln|u|\right]_1^2 = \ln 2 - \ln 1 = \ln 2.$$

17. (a) Sea $u = x + \lambda$. Se tiene

$$u = x + \lambda \quad \Longrightarrow \quad du = dx,\ u(a) = a + \lambda,\ u(b) = b + \lambda,\ x = u - \lambda.$$

Por lo tanto,

$$\int_a^b f(x)\,dx = \int_{a+\lambda}^{b+\lambda} f(u-\lambda)du = \int_{a+\lambda}^{b+\lambda} f(x-\lambda)dx.$$

La última igualdad del lado derecho es válida, ya que en una integral definida la variable de integración es una variable muda (es indistinto llamarla u o x).

(b) Sea $u = \lambda x$. Se tiene

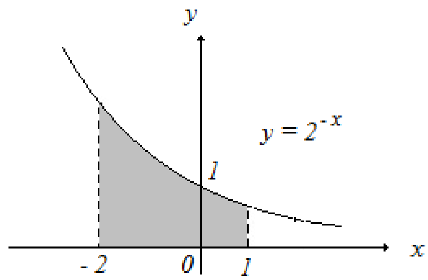
$$u = \lambda x \quad \Longrightarrow \quad du = \lambda\,dx,\ u(a) = \lambda a,\ u(b) = \lambda b,\ x = \frac{u}{\lambda}.$$

Por lo tanto,

$$\int_a^b f(x)\,dx = \frac{1}{\lambda}\int_a^b f(x)\,\lambda dx = \frac{1}{\lambda}\int_{\lambda a}^{\lambda b} f\left(\frac{u}{\lambda}\right)\,du = \frac{1}{\lambda}\int_{\lambda a}^{\lambda b} f\left(\frac{x}{\lambda}\right)\,dx.$$

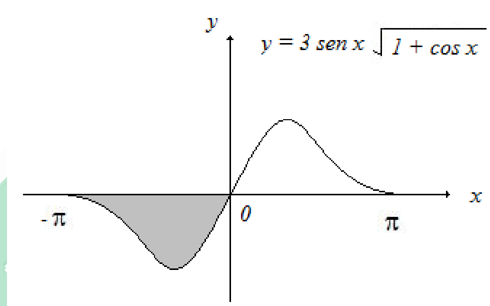
CÁLCULO III
TAREA 4 - SOLUCIONES
ÁREA. VALOR PROMEDIO. LONGITUD DE CURVA
(Tema 1.5)

1. (a) $y = 2^{-x}, \quad -2 \le x \le 1.$



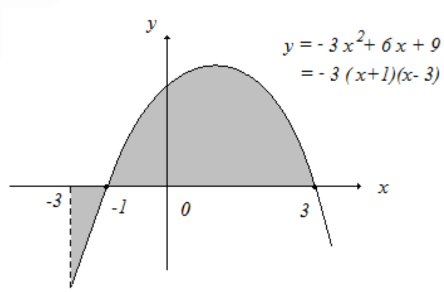
$$A = \int_{-2}^{1} 2^{-x}\, dx = -\int_{2}^{-1} 2^{u}\, du = \int_{-1}^{2} 2^{u}\, du = \frac{1}{\ln 2}\left[2^{u}\right]_{-1}^{2} = \frac{1}{\ln 2}\left[2^{2} - 2^{-1}\right]_{-1}^{2} = \frac{7}{2\ln 2}.$$

(b) $y = 3\, sen\, x\,\sqrt{1+\cos x}, \quad -\pi \le x \le 0.$



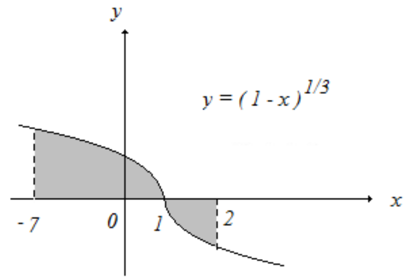
$$A = -\int_{-\pi}^{0} 3\, sen\, x\,\sqrt{1+\cos x}\, dx = 3\int_{0}^{2}\sqrt{u}\, du = 3\left(\frac{2}{3}\right)\left[u^{3/2}\right]_{0}^{2} = 2^{5/2}.$$

(c) $y = -3x^2 + 6x + 9, \quad -3 \le x \le 3.$



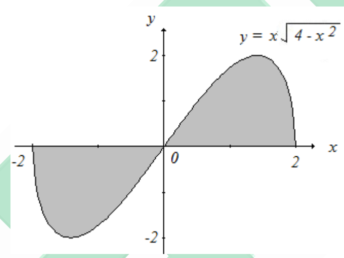
$$\begin{aligned} A &= \int_{-3}^{-1}\left(3x^2 - 6x - 9\right)\, dx + \int_{-1}^{3}\left(-3x^2 + 6x + 9\right)\, dx \\ &= \left[x^3 - 3x^2 - 9x\right]_{-3}^{-1} + \left[-x^3 + 3x^2 + 9x\right]_{-1}^{3} = 64. \end{aligned}$$

(d) $y=(1-x)^{1/3}, \quad -7\leq x\leq 2.$



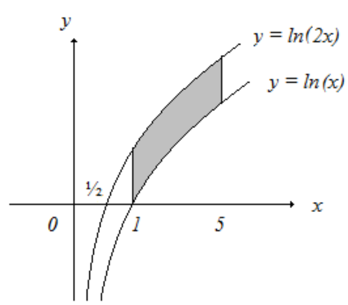
$$\begin{aligned} A &= \int_{-7}^{1}(1-x)^{1/3}\ dx-\int_{1}^{2}(1-x)^{1/3}\ dx=-\int_{8}^{0}u^{1/3}\ du+\int_{0}^{-1}u^{1/3}\ du \\ &= \int_{0}^{8}u^{1/3}\ du-\int_{-1}^{0}u^{1/3}\ du=\frac{3}{4}\left[u^{4/3}\right]_{0}^{8}-\frac{3}{4}\left[u^{4/3}\right]_{-1}^{0}=\frac{51}{4}. \end{aligned}$$

(e) $y=x\sqrt{4-x^2}, \quad -2\leq x\leq 2.$



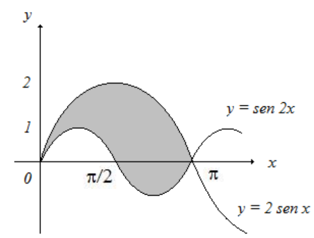
$$\begin{aligned} A &= -\int_{-2}^{0}x\sqrt{4-x^2}\ dx+\int_{0}^{2}x\sqrt{4-x^2}\ dx=\frac{1}{2}\int_{0}^{4}\sqrt{u}\ du-\frac{1}{2}\int_{4}^{0}\sqrt{u}\ du \\ &= \frac{1}{2}\int_{0}^{4}\sqrt{u}\ du+\frac{1}{2}\int_{0}^{4}\sqrt{u}\ du=\int_{0}^{4}\sqrt{u}\ du=\frac{2}{3}\left[u^{3/2}\right]_{0}^{4}=\frac{16}{3}. \end{aligned}$$

2. (a) $y=\ln x$ y $y=\ln(2x), \quad 1\leq x\leq 5.$



$$A=\int_{1}^{5}\left[\ln(2x)-\ln x\right]\ dx=\int_{1}^{5}\left[\ln 2+\ln x-\ln x\right]\ dx=\int_{1}^{5}\ \ln 2\ dx=(\ln 2)\left[x\right]_{1}^{5}=4\ln 2.$$

(b) $y = 2\,sen\,x$ y $y = sen\,(2x)\,, \quad 0 \le x \le \pi.$



$$\begin{aligned} A &= \int_0^{\pi} [2\,sen\,x - sen\,(2x)]\ dx = 2\int_0^{\pi} sen\,x\ dx - \int_0^{\pi} sen\,(2x)\ dx \\ &= 2\int_0^{\pi} sen\,x\ dx - \frac{1}{2}\int_0^{2\pi} sen\,u\ du = -2\,[\cos x]_0^{\pi} + \frac{1}{2}\,[\cos u]_0^{2\pi} \\ &= -2\left[\underbrace{\cos\pi}_{-1} - \underbrace{\cos 0}_{1}\right] + \frac{1}{2}\left[\underbrace{\cos(2\pi)}_{1} - \underbrace{\cos 0}_{1}\right] = 4. \end{aligned}$$

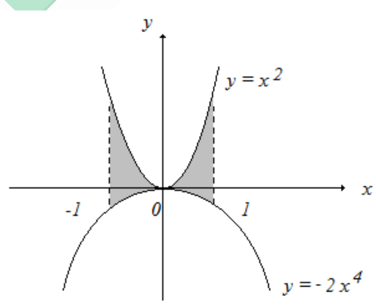
Nota: Otra manera de obtener el mismo resultado consiste en utilizar la identidad trigonométrica

$$sen\,(2x) = 2\ sen\,x\ \cos x.$$

De esta manera,

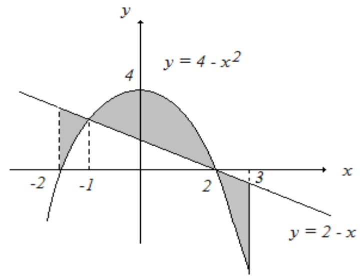
$$\begin{aligned} A &= \int_0^{\pi} [2\,sen\,x - sen\,(2x)]\ dx = \int_0^{\pi} [2\,sen\,x - 2\ sen\,x\ \cos x]\ dx \\ &= 2\int_0^{\pi} sen\,x\,[1 - \cos x]\ dx = 2\int_0^{2} u\ du = \left[u^2\right]_0^2 = 4. \end{aligned}$$

(c) $y = x^2$ y $y = -2x^4, \quad -1 \le x \le 1.$



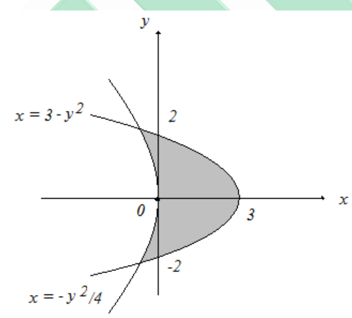
$$\begin{aligned} A &= \int_{-1}^{1} \left[x^2 - \left(-2x^4\right)\right]\ dx = \int_{-1}^{1} \left[x^2 + 2x^4\right]\ dx = 2\int_0^{1} \left[x^2 + 2x^4\right]\ dx \\ &= 2\left[\frac{x^3}{3} + \frac{2x^5}{5}\right]_0^1 = \frac{22}{15}. \end{aligned}$$

(d) $y = 4 - x^2$ y $y = 2 - x$, $-2 \leq x \leq 3$.



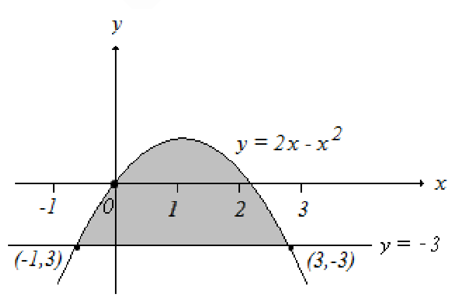
$$\begin{aligned} A &= \int_{-2}^{-1} \left[(2-x) - \left(4-x^2\right)\right] \, dx + \int_{-1}^{2} \left[\left(4-x^2\right) - (2-x)\right] \, dx + \int_{2}^{3} \left[(2-x) - \left(4-x^2\right)\right] \, dx \\ &= \int_{-2}^{-1} \left(x^2 - x - 2\right) \, dx + \int_{-1}^{2} \left(-x^2 + x + 2\right) \, dx + \int_{2}^{3} \left(x^2 - x - 2\right) \, dx \\ &= \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_{-2}^{-1} + \left[-\frac{x^3}{3} + \frac{x^2}{2} + 2x\right]_{-1}^{2} + \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_{2}^{3} = \frac{49}{6}. \end{aligned}$$

(e) $x + y^2 = 3$ y $4x + y^2 = 0$, $-2 \leq y \leq 2$.



$$A = \int_{-2}^{2} \left[\left(3 - y^2\right) - \left(-\frac{y^2}{4}\right)\right] \, dy = \int_{-2}^{2} \left[3 - \frac{3y^2}{4}\right] \, dy = \left[3y - \frac{y^3}{4}\right]_{-2}^{2} = 8.$$

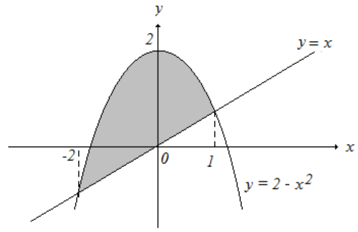
3. (a) $y = 2x - x^2$ y $y = -3$.



$$\begin{gathered} 2x - x^2 = -3 \\ x^2 - 2x - 3 = 0 \\ (x+1)(x-3) = 0 \\ \therefore \; x = -1 \text{ o } x = 3. \end{gathered}$$

$$A = \int_{-1}^{3} \left[\left(2x - x^2\right) - (-3)\right] \, dx = \int_{-1}^{3} \left[2x - x^2 + 3\right] \, dx = \left[x^2 - \frac{x^3}{3} + 3x\right]_{-1}^{3} = \frac{32}{3}.$$

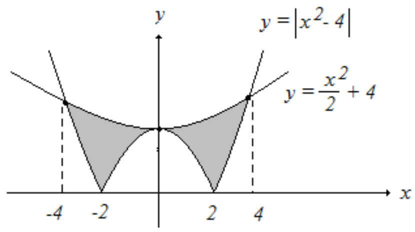
(b) $y = 2 - x^2$ y $y = x$.



$$2 - x^2 = x$$
$$x^2 + x - 2 = 0$$
$$(x+2)(x-1) = 0$$
$$\therefore\ x = -2 \text{ o } x = 1.$$

$$A = \int_{-2}^{1} \left[2 - x^2 - x\right]\, dx = \left[2x - \frac{x^3}{3} - \frac{x^2}{2}\right]_{-2}^{1} = \frac{9}{2}.$$

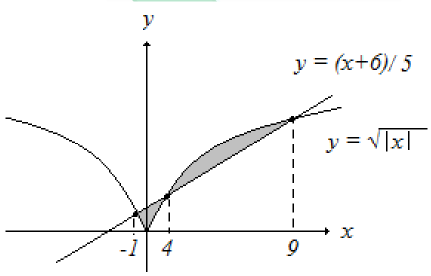
(c) $y = |x^2 - 4|$ y $y = \dfrac{x^2}{2} + 4$.



$$|x^2 - 4| = \tfrac{x^2}{2} + 4$$
$$|x^2 - 4|^2 = \left(\tfrac{x^2}{2} + 4\right)^2$$
$$x^4 - 8x^2 + 16 = \tfrac{x^4}{4} + 4x^2 + 16$$
$$3x^2\left(x^2 - 16\right) = 0$$
$$\therefore\ x = -4 \text{ o } x = 0 \text{ o } x = 4.$$

$$\begin{aligned} A &= \int_{-4}^{-2} \left[\left(\frac{x^2}{2} + 4\right) - \left(x^2 - 4\right)\right] dx + \int_{-2}^{2} \left[\left(\frac{x^2}{2} + 4\right) - \left(4 - x^2\right)\right] dx \\ &\quad + \int_{2}^{4} \left[\left(\frac{x^2}{2} + 4\right) - \left(x^2 - 4\right)\right] dx \\ &= 2\int_{0}^{2} \left[\left(\frac{x^2}{2} + 4\right) - \left(4 - x^2\right)\right] dx + 2\int_{2}^{4} \left[\left(\frac{x^2}{2} + 4\right) - \left(x^2 - 4\right)\right] dx = \frac{64}{3}. \end{aligned}$$

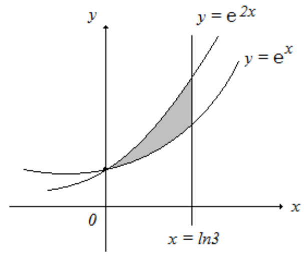
(d) $y = \sqrt{|x|}$ y $5y = x + 6$.



$$\sqrt{|x|} = \tfrac{x+6}{5},\quad x \ge -6$$
$$25\,|x| = (x+6)^2$$
i) $x \ge 0 \Longrightarrow 25x = (x+6)^2$
$$x^2 - 13x + 36 = 0$$
$$(x-4)(x-9) = 0$$
$$\therefore\ x = 4 \text{ o } x = 9.$$
ii) $-6 \le x < 0 \Longrightarrow -25x = (x+6)^2$
$$x^2 + 37x + 36 = 0$$
$$(x+1)(x+36) = 0$$
$$\therefore\ x = -1.$$

$$\begin{aligned} A &= \int_{-1}^{0} \left[\left(\frac{x}{5} + \frac{6}{5}\right) - \sqrt{-x}\right] dx + \int_{0}^{4} \left[\left(\frac{x}{5} + \frac{6}{5}\right) - \sqrt{x}\right] dx \\ &\quad + \int_{4}^{9} \left[\sqrt{x} - \left(\frac{x}{5} + \frac{6}{5}\right)\right] dx = \frac{5}{3}. \end{aligned}$$

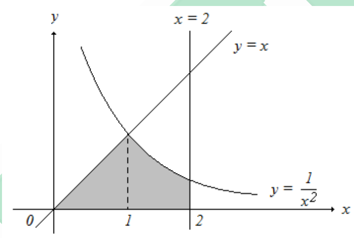
4. (a) $y = e^x$, $y = e^{2x}$ y $x = \ln 3$.



$$A = \int_{-1}^{0} \left(e^{2x} - e^x\right)\, dx = \left[\frac{1}{2}e^{2x} - e^x\right]_0^{\ln 3} = \left[\frac{1}{2}e^{2\ln 3} - e^{\ln 3}\right] - \left[\frac{1}{2} - 1\right]$$

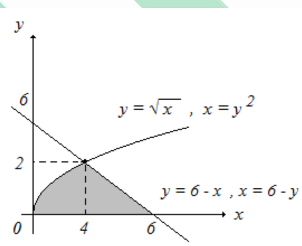
$$= \left[\frac{1}{2}e^{\ln 9} - e^{\ln 3}\right] + \frac{1}{2} = \left(\frac{9}{2} - 3\right) + \frac{1}{2} = 2.$$

(b) $y = x$, $y = \dfrac{1}{x^2}$, $x = 2$ y $y = 0$.



$$A = \int_{0}^{1} (x - 0)\, dx + \int_{1}^{2} \left(\frac{1}{x^2} - 0\right)\, dx = \left[\frac{x^2}{2}\right]_0^1 - \left[\frac{1}{x}\right]_1^2 = 1.$$

5. $y = \sqrt{x}$, $y = 6 - x$ y $y = 0$.



$$\begin{gathered} \sqrt{x} = 6 - x, \quad x \leq 6 \\ x = 36 - 12x + x^2 \\ x^2 - 13x + 36 = 0 \\ (x - 4)(x - 9) = 0 \\ \therefore\ x = 4. \end{gathered}$$

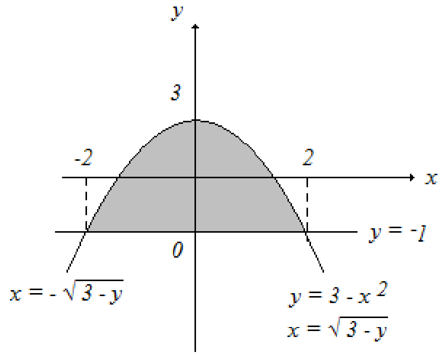
(a) Integrando con respecto a x (región y-simple):

$$A = \int_{0}^{4} \left(\sqrt{x} - 0\right)\, dx + \int_{4}^{6} [(6 - x) - 0]\, dx = \frac{22}{3}.$$

(b) Integrando con respecto a y (región x-simple):

$$A = \int_{0}^{2} \left[(6 - y) - y^2\right]\, dy = \frac{22}{3}.$$

6. $y = 3 - x^2$ y $y = -1$.



$$
\begin{aligned}
3 - x^2 &= -1 \\
x^2 &= 4 \\
|x| &= 2 \\
\therefore\ x = 2 \ \text{ o } \ x &= -2.
\end{aligned}
$$

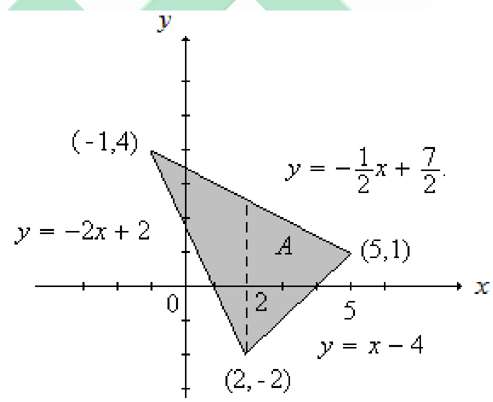
(a) Integrando con respecto a x (región y-simple):

$$A = \int_{-2}^{2} \left[\left(3 - x^2\right) - (-1)\right]\, dx = \int_{-2}^{2} \left(4 - x^2\right)\, dx = 2\int_{0}^{2} \left(4 - x^2\right)\, dx = \frac{32}{3}.$$

(b) Integrando con respecto a y (región x-simple):

$$
\begin{aligned}
A &= \int_{-1}^{3} \left[\sqrt{3-y} - \left(-\sqrt{3-y}\right)\right]\, dy = 2\int_{-1}^{3} \sqrt{3-y}\, dy = -2\int_{4}^{0} \sqrt{u}\, du \\
&= 2\int_{0}^{4} \sqrt{u}\, du = \frac{4}{3}\left[u^{3/2}\right]_0^4 = \frac{32}{3}.
\end{aligned}
$$

7. En la figura se muestra las ecuaciones de las rectas que unen a cada par de puntos:

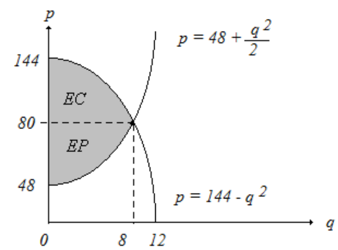


En consecuencia,

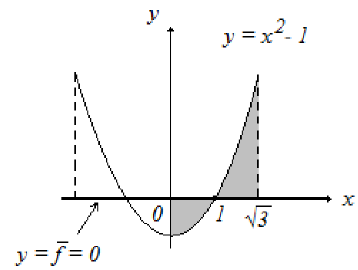
$$A = \int_{-1}^{2} \left[\left(-\frac{1}{2}x + \frac{7}{2}\right) - (-2x+2)\right]\, dx + \int_{2}^{5} \left[\left(-\frac{1}{2}x + \frac{7}{2}\right) - (x-4)\right]\, dx = \frac{27}{2}.$$

8. Los excedentes del consumidor EC y del productor EP están dados por

$$
\begin{aligned}
EC &= \int_{0}^{8} \left[\left(144 - q^2\right) - 80\right]\, dq = \frac{1024}{3} \approx 341.3, \\
EP &= \int_{0}^{8} \left[80 - \left(48 + \frac{q^2}{2}\right)\right]\, dq = \frac{512}{3} \approx 170.7 = \frac{EC}{2}.
\end{aligned}
$$



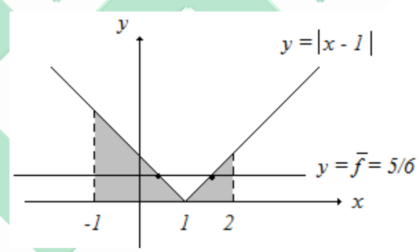
9. (a) $f(x) = x^2 - 1$ en $\left[0, \sqrt{3}\right]$.



$$\bar{f} \;=\; \frac{1}{\sqrt{3}-0}\int_0^{\sqrt{3}} \left(x^2-1\right)\,dx = \frac{1}{\sqrt{3}}\left[\frac{x^3}{3}-x\right]_0^{\sqrt{3}} = 0.$$

$$\therefore \qquad \bar{f} = 0 \;\text{ y se alcanza en } x = 1,\; \text{ya que } f(1) = 0.$$

(b) $f(x) = |x-1|$ en $[-1, 2]$.



$$\bar{f} \;=\; \frac{1}{2-(-1)}\int_{-1}^{2} |x-1|\;dx = \frac{1}{3}\left[\int_{-1}^{1}(1-x)\;dx + \int_{-1}^{2}(x-1)\;dx\right] = \frac{5}{6}.$$

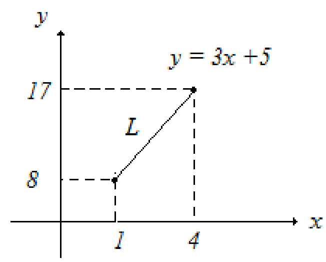
$$\therefore \qquad \bar{f} = \frac{5}{6} \;\text{ y se alcanza en } x_1 = \frac{1}{6},\; \text{y } x_2 = \frac{11}{6}, \text{ ya que } f\left(\frac{1}{6}\right) = f\left(\frac{11}{6}\right) = \frac{5}{6}.$$

10. Sea f continua y tal que $\int_1^2 f(x)\;dx = 4$. Como

$$\bar{f} = \frac{1}{2-1}\int_1^2 f(x)\;dx = \frac{1}{1}(4) = 4,$$

y como f es una función continua en $[1,2]$, por el Teorema del Valor Medio para integrales definidas existe $c \in [1,2]$ tal que $f(c) = 4$. Por lo tanto, $f(x) = 4$ al menos una vez en $[1,2]$.

11.



La longitud L del segmento de la recta $y = 3x + 5$ entre $x = 1$ y $x = 4$ está dada por

$$L = \int_1^4 \sqrt{1 + [y'(x)]^2}\, dx.$$

Como $y(x) = 3x + 5$, por lo tanto $y'(x) = 3$, de modo que

$$L = \int_1^4 \sqrt{1 + 3^2}\, dx = \sqrt{10} \int_1^4 dx = 3\sqrt{10}.$$

En efecto, la distancia L entre el punto $(1, 8)$ y el $(4, 17)$ es

$$L = \sqrt{(4-1)^2 + (17-8)^2} = \sqrt{9 + 81} = \sqrt{90} = 3\sqrt{10}.$$

12. Como $y(x) = 2x^{3/2}$, por lo tanto $y'(x) = 3x^{1/2}$, de modo que

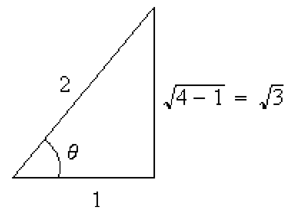
$$L = \int_{1/3}^{7} \sqrt{1 + [3x^{1/2}]^2}\, dx = \int_{1/3}^{7} \sqrt{1 + 9x}\, dx = \frac{1}{9}\int_4^{64} \sqrt{u}\, du = \frac{2}{27}\left[u^{3/2}\right]_4^{64} = \frac{112}{3}.$$

CÁLCULO III
TAREA 5 - SOLUCIONES
INTEGRALES RELACIONADAS CON LAS FUNCIONES TRIGONOMÉTRICAS INVERSAS
(Tema 1.6)

1. (a) $\csc\left(\cos^{-1}\left(\frac{1}{2}\right)\right)$.

$$\text{Sea} \quad \theta = \cos^{-1}\left(\frac{1}{2}\right)$$

$$\therefore \quad \cos\theta = \frac{1}{2} = \frac{\text{cateto adyacente}}{\text{hipotenusa}}$$

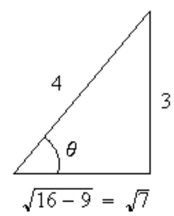


$$\therefore \quad \csc\left(\cos^{-1}\left(\frac{1}{2}\right)\right) = \csc\theta = \frac{2}{\sqrt{3}}.$$

(b) $\sec\left(sen^{-1}\left(\frac{3}{4}\right)\right)$.

$$\text{Sea} \quad \theta = sen^{-1}\left(\frac{3}{4}\right)$$

$$\therefore \quad sen\,\theta = \frac{3}{4} = \frac{\text{cateto opuesto}}{\text{hipotenusa}}$$

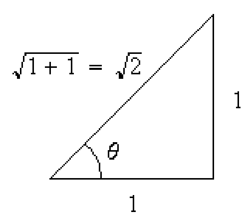


$$\therefore \quad \sec\left(sen^{-1}\left(\frac{3}{4}\right)\right) = \sec\theta = \frac{4}{\sqrt{7}}.$$

(c) $\cos\left(\tan^{-1}(1)\right)$.

$$\text{Sea} \quad \theta = \tan^{-1}(1)$$

$$\therefore \quad \tan\theta = 1 = \frac{\text{cateto opuesto}}{\text{cateto adyacente}}$$



$$\therefore \quad \cos\left(\tan^{-1}(1)\right) = \cos\theta = \frac{1}{\sqrt{2}}.$$

2.

$$\begin{aligned}
\log_5\left(2x + \tan^{-1}(y)\right) - \log_5 x &= 1 \\
\log_5\left(\frac{2x + \tan^{-1} y}{x}\right) &= 1 \\
5^{\log_5\left(\frac{2x+\tan^{-1} y}{x}\right)} &= 5^1 \\
\frac{2x + \tan^{-1} y}{x} &= 5 \\
2x + \tan^{-1} y &= 5x \\
\tan^{-1} y &= 3x \\
y &= \tan(3x).
\end{aligned}$$

3. (a) Si $y = sen^{-1}\left(\sqrt{2}\,x\right)$, entonces

$$\frac{dy}{dx} = \frac{dsen^{-1}\left(\sqrt{2}\,x\right)}{dx} = \frac{1}{\sqrt{1-\left(\sqrt{2}\,x\right)^2}}\frac{d\left(\sqrt{2}\,x\right)}{dx} = \frac{\sqrt{2}}{\sqrt{1-2\,x^2}}.$$

(b) Si $y = \cot^{-1}(1/x) - \tan^{-1} x$, entonces

$$\begin{aligned}\frac{dy}{dx} &= \frac{d\left[\cot^{-1}(1/x) - \tan^{-1} x\right]}{dx} = -\frac{1}{1+(1/x)^2}\frac{d(1/x)}{dx} - \frac{1}{1+x^2}\\ &= -\frac{1}{1+(1/x)^2}\left(-\frac{1}{x^2}\right) - \frac{1}{1+x^2} = \frac{1}{x^2+1} - \frac{1}{1+x^2} = 0.\end{aligned}$$

En otras palabras, $\frac{d\cot^{-1}(1/x)}{dx} = \frac{d\tan^{-1} x}{dx}$.

(c) Si $y = \ln\left(\tan^{-1} x\right)$, entonces

$$\frac{dy}{dx} = \frac{d\ln\left(\tan^{-1} x\right)}{dx} = \frac{1}{\tan^{-1} x}\frac{d\tan^{-1} x}{dx} = \frac{1}{\tan^{-1} x}\left(\frac{1}{1+x^2}\right) = \frac{1}{(1+x^2)\tan^{-1} x}.$$

(d) Si $y = e^{\tan^{-1}\left(x^5\right)}$, entonces

$$\frac{dy}{dx} = \frac{de^{\tan^{-1}\left(x^5\right)}}{dx} = e^{\tan^{-1}\left(x^5\right)}\frac{d\tan^{-1}\left(x^5\right)}{dx} = e^{\tan^{-1}\left(x^5\right)}\left(\frac{1}{1+\left(x^5\right)^2}\right)\frac{dx^5}{dx} = \frac{5x^4 e^{\tan^{-1}\left(x^5\right)}}{1+x^{10}}.$$

(e) Si $y = \csc^{-1}\left(e^x\right)$, entonces

$$\frac{dy}{dx} = \frac{d\csc^{-1}\left(e^x\right)}{dx} = -\frac{1}{\left|e^x\right|\sqrt{\left(e^x\right)^2-1}}\frac{de^x}{dx} = -\frac{e^x}{e^x\sqrt{e^{2x}-1}} = -\frac{1}{\sqrt{e^{2x}-1}}.$$

4. (a)

$$\int\frac{dx}{\sqrt{1-4x^2}} = \frac{1}{2}\int\frac{2\,dx}{\sqrt{1-(2x)^2}} = \frac{1}{2}\int\frac{du}{\sqrt{1-u^2}} = \frac{1}{2}sen^{-1}u + C = \frac{1}{2}sen^{-1}(2x) + C.$$

(b)

$$\begin{aligned}\int\frac{dy}{\sqrt{3+4y-4y^2}} &= \int\frac{dy}{\sqrt{4-(2y-1)^2}} = \frac{1}{2}\int\frac{2\,dy}{\sqrt{4-(2y-1)^2}}\\ &= \frac{1}{2}\int\frac{du}{\sqrt{4-u^2}} = \frac{1}{2}sen^{-1}\left(\frac{u}{2}\right) + C = \frac{1}{2}sen^{-1}\left(\frac{2y-1}{2}\right) + C.\end{aligned}$$

(c)

$$\int\frac{e^t\,dt}{\sqrt{1-e^{2t}}} = \int\frac{e^t\,dt}{\sqrt{1-\left(e^t\right)^2}} = \int\frac{du}{\sqrt{1-u^2}} = sen^{-1}u + C = sen^{-1}\left(e^t\right) + C.$$

(d) $\int \frac{dt}{7+3t^2}$

$$\begin{aligned}\int \frac{dt}{7+3t^2} &= \frac{1}{\sqrt{3}}\int \frac{\sqrt{3}\,dt}{\left(\sqrt{7}\right)^2+\left(\sqrt{3}t\right)^2} = \frac{1}{\sqrt{3}}\left[\frac{1}{\sqrt{7}}\tan^{-1}\left(\frac{\sqrt{3}t}{\sqrt{7}}\right)\right]+C \\ &= \frac{1}{\sqrt{21}}\tan^{-1}\left(\sqrt{\frac{3}{7}}t\right)+C.\end{aligned}$$

(e)

$$\int \frac{sen\,\theta\; d\theta}{1+\cos^2\theta} = -\int \frac{du}{1+u^2} = -\tan^{-1}u + C = -\tan^{-1}(\cos\theta)+C.$$

(f)

$$\begin{aligned}\int \frac{3^x\,dx}{1+3^{2x}} &= \int \frac{3^x\,dx}{1+(3^x)^2} = \frac{1}{\ln 3}\int \frac{3^x\ \ln 3\,dx}{1+(3^x)^2} = \frac{1}{\ln 3}\int \frac{du}{1+u^2} \\ &= \frac{1}{\ln 3}\tan^{-1}u + C = \frac{1}{\ln 3}\tan^{-1}(3^x)+C.\end{aligned}$$

(g)

$$\begin{aligned}\int \frac{dx}{x\sqrt{5x^2-2}} &= \int \frac{\sqrt{5}\,dx}{\sqrt{5}x\sqrt{\left(\sqrt{5}x\right)^2-2}} = \frac{1}{\sqrt{2}}\sec^{-1}\left|\frac{\sqrt{5}x}{\sqrt{2}}\right|+C \\ &= \frac{1}{\sqrt{2}}\sec^{-1}\left|\sqrt{\frac{5}{2}}x\right|+C.\end{aligned}$$

(h)

$$\int \frac{dx}{(1+x^2)\tan^{-1}x} = \int \left(\frac{1}{\tan^{-1}x}\right)\frac{dx}{1+x^2} = \int \frac{du}{u} = \ln|u|+C = \ln\left|\tan^{-1}x\right|+C.$$

(i)

$$\int \frac{e^{sen^{-1}x}\ dx}{\sqrt{1-x^2}} = \int e^{sen^{-1}x}\ \frac{dx}{\sqrt{1-x^2}} = \int e^u\,du = e^u + C = e^{sen^{-1}x}+C.$$

5. (a)

$$\begin{aligned}\int_0^1 \frac{4\,dx}{\sqrt{4-x^2}} &= 4\int_0^1 \frac{dx}{\sqrt{4-x^2}} = 4\left[sen^{-1}\left(\frac{x}{2}\right)\right]_0^1 = 4\left[sen^{-1}\left(\frac{1}{2}\right) - sen^{-1}(0)\right] \\ &= 4\left[\frac{\pi}{6}-0\right] = \frac{2\pi}{3}.\end{aligned}$$

(b)

$$\begin{aligned}\int_1^e \frac{dt}{t\left(1+\ln^2 t\right)} &= \int_1^e \frac{1}{\left(1+(\ln t)^2\right)}\frac{dt}{t} = \int_0^1 \frac{du}{1+u^2} = \left[\tan^{-1}u\right]_0^1 \\ &= \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}.\end{aligned}$$

(c)

$$\int_1^2 \frac{8\,dt}{t^2-2t+2} = 8\int_1^2 \frac{dt}{(t-1)^2+1} = 8\int_0^1 \frac{du}{u^2+1} = 8\left[\tan^{-1} u\right]_0^1$$
$$= 8\left[\tan^{-1}(1) - \tan^{-1}(0)\right] = 8\left[\frac{\pi}{4} - 0\right] = 2\pi.$$

(d) Método1:

$$\int_{\sqrt{2}}^2 \frac{\sec^2(\sec^{-1} x)\,dx}{x\sqrt{x^2-1}} = \int_{\sqrt{2}}^2 \sec^2(\sec^{-1} x)\,\frac{dx}{x\sqrt{x^2-1}} = \int_{\sec^{-1}(\sqrt{2})}^{\sec^{-1}(2)} \sec^2 u\,du$$
$$= \int_{\pi/4}^{\pi/3} \sec^2 u\,du = [\tan u]_{\pi/4}^{\pi/3} = \tan\left(\frac{\pi}{3}\right) - \tan\left(\frac{\pi}{4}\right) = \sqrt{3} - 1.$$

Método 2:
Como $\sec(\sec^{-1} x) = x$, por lo tanto $\sec^2(\sec^{-1} x) = [\sec(\sec^{-1} x)]^2 = x^2$. De esta manera,

$$\int_{\sqrt{2}}^2 \frac{\sec^2(\sec^{-1} x)\,dx}{x\sqrt{x^2-1}} = \int_{\sqrt{2}}^2 \frac{x^2\,dx}{x\sqrt{x^2-1}} = \int_{\sqrt{2}}^2 \frac{x\,dx}{\sqrt{x^2-1}} = \int_{\sqrt{2}}^2 \frac{2x\,dx}{2\sqrt{x^2-1}}$$
$$= \int_1^3 \frac{du}{2\sqrt{u}} = \left[\sqrt{u}\right]_1^3 = \sqrt{3} - 1.$$

6. Las integrales indefinidas
$\int \frac{dx}{\sqrt{1-x^2}} = sen^{-1}x + C$ y $\int \frac{dx}{\sqrt{1-x^2}} = -\cos^{-1} x + C$
son equivalentes. En efecto, si en la segunda expresión utilizamos la identidad

$$sen^{-1}x + \cos^{-1} x = \frac{\pi}{2},$$

obtenemos

$$\int \frac{dx}{\sqrt{1-x^2}} = -\int \frac{-dx}{\sqrt{1-x^2}} = -\cos^{-1} x + C = sen^{-1}x - \frac{\pi}{2} + C.$$

Observamos que las dos integrales indefinidas difieren sólo en una constante, de modo que ambas respuestas son correctas.

CÁLCULO III
TAREA 6 - SOLUCIONES
TÉCNICAS DE INTEGRACIÓN
(Tema 1.7)

1. (a) $\displaystyle\int \frac{e^{2x}\,dx}{4+9e^{2x}}$.

Sea $u = 4 + 9e^{2x}$. Se tiene

$$u = 4 + 9e^{2x} \quad \Longrightarrow \quad du = 18e^{2x}\,dx.$$

Por lo tanto,

$$\begin{aligned}\int \frac{e^{2x}\,dx}{4+9e^{2x}} &= \frac{1}{18}\int \frac{18e^{2x}\,dx}{4+9e^{2x}} = \frac{1}{18}\int \frac{du}{u} = \frac{1}{18}\ln|u| + C = \frac{1}{18}\ln\left|4+9e^{2x}\right| + C\\ &= \frac{1}{18}\ln\left(4+9e^{2x}\right) + C.\end{aligned}$$

(b) $\displaystyle\int \frac{e^{x}\,dx}{4+9e^{2x}}$.

Nota que

$$\int \frac{e^{x}\,dx}{4+9e^{2x}} = \int \frac{e^{x}\,dx}{4+(3e^{x})^2}.$$

Sea $u = 3e^{x}$. Se tiene

$$u = 3e^{x} \quad \Longrightarrow \quad du = 3e^{x}\,dx.$$

Por lo tanto,

$$\begin{aligned}\int \frac{e^{x}\,dx}{4+9e^{2x}} &= \frac{1}{3}\int \frac{3e^{x}\,dx}{4+(3e^{x})^2} = \frac{1}{3}\int \frac{du}{4+u^2} = \frac{1}{3}\left[\frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right)\right] + C\\ &= \frac{1}{6}\tan^{-1}\left(\frac{3e^{x}}{2}\right) + C.\end{aligned}$$

(c) $\displaystyle\int_2^{16} \frac{1}{2x\sqrt{\ln x}}\,dx$.

Sea $u = \ln x$. Se tiene

$$u = \ln x \quad \Longrightarrow \quad du = \frac{1}{x}\,dx,\ u(2) = \ln 2,\ u(16) = \ln 16.$$

Por lo tanto,

$$\begin{aligned}\int_2^{16} \frac{1}{2x\sqrt{\ln x}}\,dx &= \int_2^{16} \frac{1}{\sqrt{\ln x}}\left(\frac{1}{2x}\right)dx = \int_{\ln 2}^{\ln 16} \frac{du}{2\sqrt{u}} = \left[\sqrt{u}\right]_{\ln 2}^{\ln 16}\\ &= \sqrt{\ln 16} - \sqrt{\ln 2} = \sqrt{4\ln 2} - \sqrt{\ln 2} = \sqrt{\ln 2}.\end{aligned}$$

(d) $\displaystyle\int \frac{dx}{x\sqrt{1-\ln x}}$.

Sea $u = 1 - \ln x$. Se tiene

$$u = 1 - \ln x \quad \Longrightarrow \quad du = -\frac{1}{x}\, dx.$$

Por lo tanto,

$$\begin{aligned}\int \frac{dx}{x\sqrt{1-\ln x}} &= -\int \frac{1}{\sqrt{1-\ln x}}\left(-\frac{1}{x}\right) dx = -\int \frac{du}{\sqrt{u}} = -2\sqrt{u} + C \\ &= -2\sqrt{1-\ln x} + C.\end{aligned}$$

(e) $\displaystyle\int \frac{dx}{x\sqrt{1-\ln^2 x}}$.

Nota que

$$\int \frac{dx}{x\sqrt{1-\ln^2 x}} = \int \frac{1}{\sqrt{1-(\ln x)^2}}\, \frac{1}{x}\, dx.$$

Sea $u = \ln x$. Se tiene

$$u = \ln x \quad \Longrightarrow \quad du = \frac{1}{x}\, dx.$$

Por lo tanto,

$$\begin{aligned}\int \frac{dx}{x\sqrt{1-\ln^2 x}} &= \int \frac{1}{\sqrt{1-(\ln x)^2}}\left(\frac{1}{x}\right) dx = \int \frac{du}{\sqrt{1-u^2}} = sen^{-1}(u) + C \\ &= sen^{-1}(\ln x) + C.\end{aligned}$$

(f) $\displaystyle\int \frac{\ln x}{x + 4x\ln^2 x}\, dx$.

Nota que

$$\int \frac{\ln x}{x + 4x\ln^2 x}\, dx = \int \frac{\ln x}{x\left(1 + 4\ln^2 x\right)}\, dx = \int \frac{1}{1 + 4(\ln x)^2}\, \frac{\ln x}{x}\, dx.$$

Sea $u = 1 + 4(\ln x)^2$. Se tiene

$$u = 1 + 4(\ln x)^2 \quad \Longrightarrow \quad du = 8(\ln x)\, \frac{1}{x}\, dx = \frac{8\ln x}{x}\, dx.$$

Por lo tanto,

$$\begin{aligned}\int \frac{\ln x}{x + 4x\ln^2 x}\, dx &= \frac{1}{8}\int \frac{1}{1 + 4(\ln x)^2}\left(\frac{8\ln x}{x}\right) dx = \frac{1}{8}\int \frac{du}{u} = \frac{1}{8}\ln|u| + C \\ &= \frac{1}{8}\ln\left(1 + 4\ln^2 x\right) + C.\end{aligned}$$

(g) $\displaystyle\int \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\,dx.$

Sea $u = 1+\sqrt{x}$. Se tiene

$$u = 1+\sqrt{x} \quad\Longrightarrow\quad du = \frac{1}{2\sqrt{x}}\,dx.$$

Por lo tanto,

$$\begin{aligned}\int \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\,dx &= 2\int \sqrt{1+\sqrt{x}}\left(\frac{1}{2\sqrt{x}}\right)dx = 2\int \sqrt{u}\,du = \frac{4}{3}u^{3/2}+C\\ &= \frac{4}{3}\left(1+\sqrt{x}\right)^{3/2}+C.\end{aligned}$$

(h) $\displaystyle\int \frac{\sqrt{x}}{1+\sqrt{x}}\,dx.$

Sea $u = 1+\sqrt{x}$, de modo que $\sqrt{x} = u-1$. Se tiene

$$u = 1+\sqrt{x} \quad\Longrightarrow\quad du = \frac{1}{2\sqrt{x}}\,dx \quad\Longrightarrow\quad dx = 2\sqrt{x}\,du = 2\,(u-1)\;du.$$

Por lo tanto,

$$\begin{aligned}\int \frac{\sqrt{x}}{1+\sqrt{x}}\,dx &= 2\int \frac{(u-1)\,(u-1)}{u}\,du = 2\int \left(\frac{u^2-2u+1}{u}\right)du\\ &= 2\int\left(u-2+\frac{1}{u}\right)du = 2\left[\frac{u^2}{2}-2u+\ln|u|\right]+C\\ &= \left(1+\sqrt{x}\right)^2-4\left(1+\sqrt{x}\right)+2\ln\left|1+\sqrt{x}\right|+C.\\ &= \left(1+\sqrt{x}\right)^2-4\left(1+\sqrt{x}\right)+2\ln\left(1+\sqrt{x}\right)+C.\end{aligned}$$

(i) $\displaystyle\int \frac{dx}{\sqrt{x}+4x\sqrt{x}}.$

Nota que

$$\int \frac{dx}{\sqrt{x}+4x\sqrt{x}} = \int \frac{dx}{\sqrt{x}\,(1+4x)} = \int \frac{1}{1+\left(2\sqrt{x}\right)^2}\,\frac{1}{\sqrt{x}}\,dx.$$

Sea $u = 2\sqrt{x}$. Se tiene

$$u = 2\sqrt{x} \quad\Longrightarrow\quad du = \frac{1}{\sqrt{x}}\,dx.$$

Por lo tanto,

$$\begin{aligned}\int \frac{dx}{\sqrt{x}+4x\sqrt{x}} &= \int \frac{1}{1+\left(2\sqrt{x}\right)^2}\left(\frac{1}{\sqrt{x}}\right)dx = \int\frac{1}{1+u^2}\,du = \tan^{-1}(u)+C\\ &= \tan^{-1}\left(2\sqrt{x}\right)+C.\end{aligned}$$

2. (a) $\displaystyle\int \frac{x}{x-1}\,dx.$

$$\int \frac{x}{x-1}\,dx = \int \frac{x-1+1}{x-1}\,dx = \int \left(1+\frac{1}{x-1}\right)\,dx = x + \ln|x-1| + C.$$

(b) $\displaystyle\int \frac{\sqrt{x}}{x+1}\,dx.$

Sea $u=\sqrt{x}$. Se tiene

$$u=\sqrt{x} \quad\Longrightarrow\quad u^2 = x \quad\Longrightarrow\quad 2u\,du = dx.$$

Por lo tanto,

$$\begin{aligned}\int \frac{\sqrt{x}}{x+1}\,dx &= \int \frac{u}{u^2+1}\,2u\,du = 2\int \frac{u^2}{u^2+1}\,du = 2\int \frac{u^2+1-1}{u^2+1}\,du \\ &= 2\int \left(1-\frac{1}{u^2+1}\right)\,du = 2\left[u-\tan^{-1}u\right]+C \\ &= 2\sqrt{x}-2\tan^{-1}\left(\sqrt{x}\right)+C.\end{aligned}$$

(c) $\displaystyle\int \sqrt{\frac{x-1}{x^5}}\,dx.$

$$\int \sqrt{\frac{x-1}{x^5}}\,dx = \int \frac{1}{x^2}\sqrt{\frac{x-1}{x}}\,dx = \int \frac{1}{x^2}\sqrt{1-\frac{1}{x}}\,dx.$$

Sea $u = 1-\dfrac{1}{x}$. Se tiene

$$u = 1-\frac{1}{x} \quad\Longrightarrow\quad du = \frac{1}{x^2}\,dx.$$

Por lo tanto,

$$\int \sqrt{\frac{x-1}{x^5}}\,dx = \int \sqrt{1-\frac{1}{x}}\left(\frac{1}{x^2}\right)\,dx = \int \sqrt{u}\,du = \frac{2}{3}u^{3/2}+C = \frac{2}{3}\left(1-\frac{1}{x}\right)^{3/2}+C.$$

(d) $\displaystyle\int (\sec x + \cot x)^2\,dx.$

$$\begin{aligned}\int (\sec x+\cot x)^2\,dx &= \int \left(\sec^2 x + 2\sec x\cot x + \cot^2 x\right)\,dx \\ &= \int \left[\sec^2 x + 2\left(\frac{1}{\cos x}\right)\left(\frac{\cos x}{sen\, x}\right) + \left(\csc^2 x - 1\right)\right] dx \\ &= \int \left(\sec^2 x + 2\csc x + \csc^2 x - 1\right)\,dx \\ &= \tan x - 2\ln|\csc x + \cot x| - \cot x - x + C.\end{aligned}$$

(e) $\displaystyle\int \frac{1}{1+sen\,x}\,dx.$

$$\begin{aligned}\int \frac{1}{1+sen\,x}\,dx &= \int \frac{1}{1+sen\,x}\left(\frac{1-sen\,x}{1-sen\,x}\right)dx = \int \frac{1-sen\,x}{1-sen^2 x}\,dx = \int \frac{1-sen\,x}{\cos^2 x}\,dx \\ &= \int \frac{1}{\cos^2 x}\,dx - \int \frac{sen\,x}{\cos^2 x}d\,x = \int \sec^2 x\,dx - \int \sec x \tan x\,dx \\ &= \tan x - \sec x + C.\end{aligned}$$

(f) $\displaystyle\int \frac{1-sen\,\theta}{\cos\theta}\,d\theta.$

$$\begin{aligned}\int \frac{1-sen\,\theta}{\cos\theta}\,d\theta &= \int \frac{1-sen\,\theta}{\cos\theta}\left(\frac{1+sen\,\theta}{1+sen\,\theta}\right)d\theta = \int \frac{1-sen^2\theta}{\cos\theta\,(1+sen\,\theta)}\,d\theta \\ &= \int \frac{\cos^2\theta}{\cos\theta\,(1+sen\,\theta)}\,d\theta = \int \frac{\cos\theta}{1+sen\,\theta}\,d\theta.\end{aligned}$$

Sea $u = 1 + sen\,\theta$. Se tiene

$$u = 1 + sen\,\theta \quad\Longrightarrow\quad du = \cos\theta\,d\theta.$$

Por lo tanto,

$$\int \frac{1-sen\,\theta}{\cos\theta}\,d\theta = \int \frac{\cos\theta}{1+sen\,\theta}\,d\theta = \int \frac{du}{u} = \ln|u| + C = \ln|1+sen\,\theta| + C.$$

(g) $\displaystyle\int \frac{1}{e^x+1}\,dx.$

$$\int \frac{1}{e^x+1}\,dx = \int \frac{1}{e^x+1}\left(\frac{e^{-x}}{e^{-x}}\right)dx = \int \frac{e^{-x}}{e^{-x}\,(e^x+1)}\,dx = \int \frac{e^{-x}}{1+e^{-x}}\,dx.$$

Sea $u = e^{-x}$. Se tiene

$$u = e^{-x} \quad\Longrightarrow\quad du = -e^{-x}\,dx.$$

Por lo tanto,

$$\begin{aligned}\int \frac{1}{e^x+1}\,dx &= \int \frac{e^{-x}}{1+e^{-x}}\,dx = -\int \frac{(-e^{-x})\,dx}{1+e^{-x}} = -\int \frac{du}{u} \\ &= -\ln|u| + C = -\ln\left|1+e^{-x}\right| + C = -\ln\left(1+e^{-x}\right) + C.\end{aligned}$$

(h) $\displaystyle\int \frac{1}{e^x+e^{-x}}\,dx.$

$$\int \frac{1}{e^x+e^{-x}}\,dx = \int \frac{1}{e^x+e^{-x}}\left(\frac{e^x}{e^x}\right)dx = \int \frac{e^x}{e^x\,(e^x+e^{-x})}\,dx = \int \frac{e^x}{e^{2x}+1}\,dx.$$

Sea $u = e^x$. Se tiene

$$u = e^x \quad\Longrightarrow\quad du = e^x\,dx.$$

Por lo tanto,

$$\begin{aligned}\int \frac{1}{e^x+e^{-x}}\,dx &= \int \frac{e^x}{e^{2x}+1}\,dx = \int \frac{e^x\,dx}{(e^x)^2+1} = \int \frac{du}{u^2+1}\\ &= \tan^{-1}(u)+C = \tan^{-1}(e^x)+C.\end{aligned}$$

(i) $\displaystyle\int \frac{e^x}{e^x+e^{-x}}\,dx.$

$$\int \frac{e^x}{e^x+e^{-x}}\,dx = \int \frac{e^x}{e^x+e^{-x}}\left(\frac{e^x}{e^x}\right)dx = \int \frac{e^{2x}}{e^x\left(e^x+e^{-x}\right)}\,dx = \int \frac{e^{2x}}{e^{2x}+1}\,dx.$$

Sea $u=e^{2x}$. Se tiene

$$u=e^{2x} \quad\Longrightarrow\quad du=2e^{2x}\,dx.$$

Por lo tanto,

$$\begin{aligned}\int \frac{e^x}{e^x+e^{-x}}\,dx &= \int \frac{e^{2x}}{e^{2x}+1}\,dx = \frac{1}{2}\int \frac{\left(2e^{2x}\right)\,dx}{e^{2x}+1} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u|+C\\ &= \frac{1}{2}\ln\left|e^{2x}+1\right|+C = \frac{1}{2}\ln\left(e^{2x}+1\right)+C.\end{aligned}$$

(j) $\displaystyle\int \frac{dx}{\sqrt{e^{2x}-1}}.$

$$\int \frac{dx}{\sqrt{e^{2x}-1}} = \int \frac{dx}{\sqrt{e^{2x}-1}}\left(\frac{e^x}{e^x}\right)dx = \int \frac{e^x\,dx}{e^x\sqrt{e^{2x}-1}}.$$

Sea $u=e^x$. Se tiene

$$u=e^x \quad\Longrightarrow\quad du=e^x\,dx.$$

Por lo tanto,

$$\begin{aligned}\int \frac{dx}{\sqrt{e^{2x}-1}} &= \int \frac{e^x\,dx}{e^x\sqrt{e^{2x}-1}} = \int \frac{e^x\,dx}{e^x\sqrt{(e^x)^2-1}} = \int \frac{du}{u\sqrt{u^2-1}}\\ &= \sec^{-1}|u|+C = \sec^{-1}|e^x|+C = \sec^{-1}(e^x)+C.\end{aligned}$$

(k) $\displaystyle\frac{1}{2}\int \sqrt{e^x-1}\,dx.$

Sea $u=\sqrt{e^x-1}$. Se tiene

$$u=\sqrt{e^x-1} \quad\Longrightarrow\quad x=\ln(u^2+1) \quad\Longrightarrow\quad dx=\frac{2u}{u^2+1}\,du.$$

Por lo tanto,

$$\begin{aligned}\frac{1}{2}\int \sqrt{e^x-1}\,dx &= \frac{1}{2}\int u\,\frac{2u}{u^2+1}\,du = \int \frac{u^2}{u^2+1}\,du = \int \frac{u^2+1-1}{u^2+1}\,du\\ &= \int\left(1-\frac{1}{u^2+1}\right)du = u-\tan^{-1}(u)+C\\ &= \sqrt{e^x-1}-\tan^{-1}\left(\sqrt{e^x-1}\right)+C.\end{aligned}$$

(l) $\int \frac{8}{t^2-2t+2}\,dt.$

Completando cuadrados, se obtiene

$$\int \frac{8}{t^2-2t+2}\,dt = 8\int \frac{dt}{(t-1)^2+1}.$$

Sea $u = t-1$. Se tiene

$$u = t-1 \quad\Longrightarrow\quad du = dt.$$

Por lo tanto,

$$\int \frac{8}{t^2-2t+2}\,dt = 8\int \frac{dt}{(t-1)^2+1} = 8\int \frac{du}{u^2+1} = 8\tan^{-1}(u)+C = 8\tan^{-1}(t-1)+C.$$

(m) $\int \frac{dz}{\sqrt{3-2z-z^2}}.$

Completando cuadrados, se obtiene

$$\int \frac{dz}{\sqrt{3-2z-z^2}} = \int \frac{dz}{\sqrt{4-(z+1)^2}}.$$

Sea $u = z+1$. Se tiene

$$u = z+1 \quad\Longrightarrow\quad du = dz.$$

Por lo tanto,

$$\int \frac{dz}{\sqrt{3-2z-z^2}} = \int \frac{dz}{\sqrt{4-(z+1)^2}} = \int \frac{du}{\sqrt{4-u^2}} = sen^{-1}\left(\frac{u}{2}\right)+C = sen^{-1}\left(\frac{z+1}{2}\right)+C.$$

3. (a) $\int x\, sen\,(3x)\,dx.$

Proponemos

$$\begin{aligned} u &= x \quad\Longrightarrow\quad du = dx \\ dv &= sen\,(3x)\,dx \quad\Longrightarrow\quad v = -\frac{1}{3}\cos(3x). \end{aligned}$$

Por lo tanto,

$$\begin{aligned} \int x\, sen\,(3x)\,dx &= -\frac{x}{3}\cos(3x) - \int\left(-\frac{1}{3}\cos(3x)\right)dx \\ &= -\frac{x}{3}\cos(3x) + \frac{1}{3}\int \cos(3x)\,dx \\ &= -\frac{x}{3}\cos(3x) + \frac{1}{9}sen\,(3x) + C. \end{aligned}$$

(b) $\int\limits_0^2 xe^{-x/2}\,dx.$

Primero conviene determinar $\displaystyle\int xe^{-x/2}\,dx$. Para ello, proponemos

$$\begin{aligned} u &= x \quad \Longrightarrow \quad du = dx \\ dv &= e^{-x/2}\,dx \quad \Longrightarrow \quad v = -2e^{-x/2}. \end{aligned}$$

De esta manera,

$$\begin{aligned} \int xe^{-x/2}\,dx &= -2xe^{-x/2} - \int \left(-2e^{-x/2}\right) dx = -2xe^{-x/2} + 2\int e^{-x/2}\,dx \\ &= -2xe^{-x/2} + 2\left(-2e^{-x/2}\right) + C = -2xe^{-x/2} - 4e^{-x/2} + C \\ &= -2(x+2)e^{-x/2} + C. \end{aligned}$$

Por lo tanto,

$$\int\limits_0^2 xe^{-x/2}\,dx = -2\left[(x+2)e^{-x/2}\right]_0^2 = -2\left[4e^{-1} - 2e^0\right] = -8e^{-1} + 4.$$

(c) $\displaystyle\int \frac{\ln x}{x^2}\,dx.$

Proponemos

$$\begin{aligned} u &= \ln x \quad \Longrightarrow \quad du = \frac{1}{x}\,dx \\ dv &= \frac{1}{x^2}\,dx \quad \Longrightarrow \quad v = -\frac{1}{x}. \end{aligned}$$

Por lo tanto,

$$\begin{aligned} \int \frac{\ln x}{x^2}\,dx &= -\frac{1}{x}\ln x - \int \left(-\frac{1}{x}\right)\left(\frac{1}{x}\right)\,dx = -\frac{1}{x}\ln x + \int \frac{1}{x^2}dx \\ &= -\frac{1}{x}\ln x - \frac{1}{x} + C = -\frac{1}{x}\left(\ln x + 1\right) + C. \end{aligned}$$

(d) $\int\limits_1^e \ln\sqrt{x}\,dx.$

Nota que

$$\int\limits_1^e \ln\sqrt{x}\,dx = \int\limits_1^e \ln x^{1/2}\,dx = \frac{1}{2}\int\limits_1^e \ln x\,dx.$$

Primero determinemos $\int \ln x\,dx$. Para ello, proponemos

$$\begin{aligned} u &= \ln x \quad \Longrightarrow \quad du = \frac{1}{x}\,dx \\ dv &= dx \quad \Longrightarrow \quad v = x. \end{aligned}$$

De esta manera,

$$\int \ln x\, dx = x\ln x - \int (x)\,\frac{1}{x}\, dx = x\ln x - \int dx = x\ln x - x + C.$$

Por lo tanto,

$$\begin{aligned}\int_1^e \ln\sqrt{x}\, dx &= \frac{1}{2}\int_1^e \ln x\, dx = \frac{1}{2}\left[x\ln x - x\right]_1^e = \frac{1}{2}\left[(e\ln e - e) - (\ln 1 - 1)\right] \\ &= \frac{1}{2}\left[(e-e)+1\right] = \frac{1}{2}.\end{aligned}$$

(e) $\int_1^e x^2 \ln x\, dx.$

Primero determinemos $\int x^2 \ln x\, dx$. Para ello, proponemos

$$\begin{aligned} u &= \ln x \quad \Longrightarrow \quad du = \frac{1}{x}\, dx \\ dv &= x^2\, dx \quad \Longrightarrow \quad v = \frac{x^3}{3}.\end{aligned}$$

De esta manera,

$$\int x^2 \ln x\, dx = \frac{x^3}{3}\ln x - \int\left(\frac{x^3}{3}\right)\frac{1}{x}\, dx = \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2\, dx = \frac{x^3}{3}\ln x - \frac{x^3}{9} + C.$$

Por lo tanto,

$$\begin{aligned}\int_1^e x^2\ln x\, dx &= \left[\frac{x^3}{3}\ln x - \frac{x^3}{9}\right]_1^e = \left(\frac{e^3}{3}\ln e - \frac{e^3}{9}\right) - \left(\frac{1}{3}\ln 1 - \frac{1}{9}\right) \\ &= \left(\frac{e^3}{3} - \frac{e^3}{9}\right) + \frac{1}{9} = \frac{2}{9}e^3 + \frac{1}{9}.\end{aligned}$$

(f) $\int \cos x\ \ln(sen x)\ dx.$

Proponemos

$$\begin{aligned} u &= \ln(sen\, x) \quad \Longrightarrow \quad du = \frac{\cos x}{sen\, x}\, dx \\ dv &= \cos x\, dx \quad \Longrightarrow \quad v = sen\, x.\end{aligned}$$

Por lo tanto,

$$\begin{aligned}\int \cos x\ \ln(sen x)\ dx &= sen\, x\ \ln(sen\, x) - \int (sen\, x)\left(\frac{\cos x}{sen\, x}\right) dx \\ &= sen\, x\ \ln(sen\, x) - \int \cos x\, dx \\ &= sen\, x\ \ln(sen\, x) - sen\, x + C.\end{aligned}$$

(g) $\displaystyle\int \frac{x\,dx}{\sqrt{x+1}}$.

Proponemos

$$\begin{aligned} u &= x \quad \Longrightarrow \quad du = dx \\ dv &= \frac{dx}{\sqrt{x+1}} \quad \Longrightarrow \quad v = 2\sqrt{x+1}. \end{aligned}$$

Por lo tanto,

$$\int \frac{x\,dx}{\sqrt{x+1}} = 2x\sqrt{x+1} - 2\int \sqrt{x+1}dx = 2x\sqrt{x+1} - \frac{4}{3}(x+1)^{3/2} + C.$$

(h) $\displaystyle\int \frac{e^{2x}\,dx}{\sqrt{1-e^x}}$.

Como

$$\int \frac{e^{2x}\,dx}{\sqrt{1-e^x}} = \int e^x \frac{e^x}{\sqrt{1-e^x}}\,dx,$$

proponemos

$$\begin{aligned} u &= e^x \quad \Longrightarrow \quad du = e^x\,dx \\ dv &= \frac{e^x}{\sqrt{1-e^x}}\,dx \quad \Longrightarrow \quad v = -2\sqrt{1-e^x}. \end{aligned}$$

Por lo tanto,

$$\begin{aligned} \int \frac{e^{2x}\,dx}{\sqrt{1-e^x}} &= -2e^x\sqrt{1-e^x} - \int \left(-2\sqrt{1-e^x}\right)(e^x)\,dx \\ &= -2e^x\sqrt{1-e^x} + 2\int e^x\sqrt{1-e^x}dx \\ &= -2e^x\sqrt{1-e^x} - \frac{4}{3}(1-e^x)^{3/2} + C. \end{aligned}$$

(i) $\displaystyle\int \tan^{-1}(x)\ dx$.

Proponemos

$$\begin{aligned} u &= \tan^{-1}(x) \quad \Longrightarrow \quad du = \frac{dx}{1+x^2} \\ dv &= dx \quad \Longrightarrow \quad v = x. \end{aligned}$$

Por lo tanto,

$$\int \tan^{-1}(x)\ dx = x\tan^{-1}(x) - \int \frac{x}{1+x^2}dx = x\tan^{-1}(x) - \frac{1}{2}\ln\left(1+x^2\right) + C.$$

(j) $\displaystyle\int x^2 e^{-x/2}\, dx.$

Proponemos

$$\begin{aligned} u &= x^2 &\implies& \quad du = 2x\, dx \\ dv &= e^{-x/2}\, dx &\implies& \quad v = -2e^{-x/2}. \end{aligned}$$

Por lo tanto,

$$\int x^2 e^{-x/2}\, dx = -2x^2 e^{-x/2} - \int \left(-2e^{-x/2}\right)(2x)\, dx = -2x^2 e^{-x/2} + 4 \int x e^{-x/2}\, dx.$$

Ahora integramos por partes la nueva integral. Para ello, proponemos

$$\begin{aligned} u &= x &\implies& \quad du = dx \\ dv &= e^{-x/2}\, dx &\implies& \quad v = -2e^{-x/2}. \end{aligned}$$

Por lo tanto,

$$\begin{aligned} \int x^2 e^{-x/2}\, dx &= -2x^2 e^{-x/2} + 4 \int x e^{-x/2}\, dx \\ &= -2x^2 e^{-x/2} + 4 \left[-2x e^{-x/2} - \int \left(-2e^{-x/2}\right) dx\right] \\ &= -2x^2 e^{-x/2} - 8x e^{-x/2} + 8 \int e^{-x/2} dx \\ &= -2x^2 e^{-x/2} - 8x e^{-x/2} + 8\left(-2e^{-x/2}\right) + C \\ &= -2x^2 e^{-x/2} - 8x e^{-x/2} - 16 e^{-x/2} + C \\ &= -2\left(x^2 + 4x + 8\right) e^{-x/2} + C. \end{aligned}$$

(k) $\displaystyle\int x^3 e^{x^2}\, dx.$

Como

$$\int x^3 e^{x^2}\, dx = \int x^2 \left(x e^{x^2}\right)\, dx,$$

proponemos

$$\begin{aligned} u &= x^2 &\implies& \quad du = 2x\, dx \\ dv &= x e^{x^2}\, dx &\implies& \quad v = \frac{1}{2} e^{x^2}. \end{aligned}$$

Por lo tanto,

$$\begin{aligned} \int x^3 e^{x^2}\, dx &= \frac{1}{2} x^2 e^{x^2} - \int \left(\frac{1}{2} e^{x^2}\right)(2x)\, dx = \frac{1}{2} x^2 e^{x^2} - \int x e^{x^2} dx \\ &= \frac{1}{2} x^2 e^{x^2} - \frac{1}{2} e^{x^2} + C = \frac{1}{2}\left(x^2 - 1\right) e^{x^2} + C. \end{aligned}$$

(l) $\displaystyle\int e^{-x}\cos x\,dx.$

Proponemos

$$\begin{array}{rcl} u & = & e^{-x} \quad \Longrightarrow \quad du = -e^{-x}\,dx \\ dv & = & \cos x\,dx \quad \Longrightarrow \quad v = sen\,x. \end{array}$$

Por lo tanto,

$$\begin{aligned} \int e^{-x}\cos x\,dx &= e^{-x}sen\,x - \int (sen\,x)\left(-e^{-x}\right)dx \\ &= e^{-x}sen\,x + \int e^{-x}sen\,x\,dx. \end{aligned}$$

Ahora integramos por partes la nueva integral. Para ello, proponemos

$$\begin{array}{rcl} u & = & e^{-x} \quad \Longrightarrow \quad du = -e^{-x}\,dx \\ dv & = & sen\,x\,dx \quad \Longrightarrow \quad v = -\cos x. \end{array}$$

Por lo tanto,

$$\begin{aligned} \int e^{-x}\cos x\,dx &= e^{-x}sen\,x + \int e^{-x}sen\,x\,dx \\ &= e^{-x}sen\,x + \left[-e^{-x}\cos x - \int (-\cos x)\left(-e^{-x}\right)dx\right] \\ &= e^{-x}sen\,x - e^{-x}\cos x - \int e^{-x}\cos x\,dx. \end{aligned}$$

Por lo tanto,

$$2\int e^{-x}\cos x\,dx = e^{-x}\left(sen\,x - \cos x\right),$$

de modo que

$$\int e^{-x}\cos x\,dx = \frac{1}{2}e^{-x}\left(sen\,x - \cos x\right) + C.$$

(m) $\displaystyle\int sen\,(\ln x)\,dx.$

Proponemos

$$\begin{array}{rcl} u & = & sen\,(\ln x) \quad \Longrightarrow \quad du = \cos(\ln x)\left(\dfrac{1}{x}\right)dx \\ dv & = & dx \quad \Longrightarrow \quad v = x. \end{array}$$

Por lo tanto,

$$\int sen\,(\ln x)\,dx = x sen\,(\ln x) - \int \cos(\ln x)\,dx.$$

Ahora integramos por partes la nueva integral. Para ello, proponemos

$$\begin{aligned} u &= \cos(\ln x) \quad \Longrightarrow \quad du = -sen\,(\ln x)\left(\frac{1}{x}\right)\,dx \\ dv &= dx \quad \Longrightarrow \quad v = x. \end{aligned}$$

Por lo tanto,

$$\begin{aligned} \int sen\,(\ln x)\,dx &= x sen\,(\ln x) - \int \cos(\ln x)\,dx \\ &= x sen\,(\ln x) - \left[x\cos(\ln x) - \int (-sen\,(\ln x))\,dx\right] \\ &= x\,[sen\,(\ln x) - \cos(\ln x)] - \int sen\,(\ln x)\,dx. \end{aligned}$$

De esta manera,

$$2\int sen\,(\ln x)\,dx = x\,[sen\,(\ln x) - \cos(\ln x)]\,,$$

de modo que

$$\int sen\,(\ln x)\,dx = \frac{x}{2}\,[sen\,(\ln x) - \cos(\ln x)] + C.$$

(n) $\int e^{\sqrt{x}}\,dx.$

Primero proponemos la sustitución $y = \sqrt{x}$. Se tiene

$$y = \sqrt{x} \quad \Longrightarrow \quad y^2 = x \quad \Longrightarrow \quad 2y\,dy = dx.$$

Por lo tanto,

$$\int e^{\sqrt{x}}\,dx = \int e^y\,(2y)\,dy = 2\int ye^y\,dy.$$

Esta última se integra por partes. Para ello, proponemos

$$\begin{aligned} u &= y \quad \Longrightarrow \quad du = dy \\ dv &= e^y\,dy \quad \Longrightarrow \quad v = e^y. \end{aligned}$$

Por lo tanto,

$$\begin{aligned} \int e^{\sqrt{x}}\,dx &= 2\int ye^y\,dy = 2\left[ye^y - \int e^y\,dy\right] = 2\,[ye^y - e^y] + C \\ &= 2\,(y-1)\,e^y + C = 2\left(\sqrt{x}-1\right)e^{\sqrt{x}} + C. \end{aligned}$$

(o) $\int \cos\left(\sqrt{x}\right)\ dx.$

Primero proponemos la sustitución $y = \sqrt{x}$. Se tiene

$$y = \sqrt{x} \quad \Longrightarrow \quad x = y^2 \quad \Longrightarrow \quad dx = 2y\ dy.$$

Por lo tanto,

$$\int \cos\left(\sqrt{x}\right)\ dx = 2\int y\cos y\ dy.$$

Esta última se integra por partes. Para ello, proponemos

$$\begin{aligned} u &= y \quad \Longrightarrow \quad du = dy \\ dv &= \cos y\ dy \quad \Longrightarrow \quad v = sen\, y. \end{aligned}$$

Por lo tanto,

$$\int \cos\left(\sqrt{x}\right)\ dx = 2\int y\cos y\ dy = 2\left[y\ sen\, y - \int sen\, y\ dy\right] = 2y\ sen\, y + 2\cos y + C.$$

4. (a) Hay que demostrar $\int x^n e^{ax}\ dx = \dfrac{x^n e^{ax}}{a} - \dfrac{n}{a}\int x^{n-1}e^{ax}\ dx$, para toda $a \neq 0$.

Proponemos

$$\begin{aligned} u &= x^n \quad \Longrightarrow \quad du = nx^{n-1}\ dx \\ dv &= e^{ax}\ dx \quad \Longrightarrow \quad v = \frac{1}{a}e^{ax}. \end{aligned}$$

Por lo tanto,

$$\int x^n e^{ax}\ dx = \frac{x^n e^{ax}}{a} - \int \left(\frac{1}{a}e^{ax}\right)\left(nx^{n-1}\right)\ dx = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\ dx.$$

(b)

$$\begin{aligned} \int x^3 e^x\ dx &= x^3e^x - 3\int x^2 e^x\ dx = x^3e^x - 3\left(x^2e^x - 2\int xe^x\ dx\right) \\ &= x^3e^x - 3x^2e^x + 6\int xe^x\ dx = x^3e^x - 3x^2e^x + 6\left(xe^x - \int e^x\ dx\right) \\ &= x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C. \end{aligned}$$

5. (a) $\displaystyle\int \frac{x+4}{x^2+5x+6}\ dx.$

Proponemos

$$\begin{aligned} \frac{x+4}{x^2+5x+6} &= \frac{x+4}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3} \\ &= \frac{A(x+3) + B(x+2)}{(x+2)(x+3)} = \frac{(A+B)x + (3A+2B)}{(x+2)(x+3)}, \end{aligned}$$

de donde

$$\begin{aligned} A+B &= 1 \\ 3A+2B &= 4. \end{aligned}$$

Al resolver el sistema de ecuaciones se obtiene

$$A = 2 \text{ y } B = -1,$$

por lo que

$$\frac{x+4}{x^2+5x+6} = \frac{2}{x+2} - \frac{1}{x+3}.$$

Por lo tanto,

$$\begin{aligned} \int \frac{x+4}{x^2+5x+6}\,dx &= \int \left(\frac{2}{x+2} - \frac{1}{x+3}\right) dx = 2\int \frac{dx}{x+2} - \int \frac{dx}{x+3} \\ &= 2\ln|x+2| - \ln|x+3| + C. \end{aligned}$$

(b) $\displaystyle\int \frac{1}{(x+1)(x^2+1)}\,dx.$

Proponemos

$$\begin{aligned} \frac{1}{(x+1)(x^2+1)} &= \frac{A}{x+1} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1)+(Bx+C)(x+1)}{(x+1)(x^2+1)} \\ &= \frac{(A+B)x^2+(B+C)x+(A+C)}{(x+1)(x^2+1)}, \end{aligned}$$

de donde

$$\begin{aligned} A+B &= 0 \\ B+C &= 0 \\ A+C &= 1. \end{aligned}$$

Al resolver el sistema de ecuaciones se obtiene

$$A = C = 1/2 \text{ y } B = -1/2,$$

por lo que

$$\frac{1}{(x+1)(x^2+1)} = \frac{1/2}{x+1} + \frac{-\frac{1}{2}x+\frac{1}{2}}{x^2+1}.$$

Por lo tanto,

$$\begin{aligned} \int \frac{1}{(x+1)(x^2+1)}\,dx &= \int \left(\frac{1/2}{x+1} + \frac{-\frac{1}{2}x+\frac{1}{2}}{x^2+1}\right) dx \\ &= \frac{1}{2}\int \frac{dx}{x+1} - \frac{1}{2}\int \frac{x}{x^2+1}dx + \frac{1}{2}\int \frac{dx}{x^2+1} \\ &= \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln\left|x^2+1\right| + \frac{1}{2}\tan^{-1}(x) + C. \end{aligned}$$

(c) $\int \frac{x}{(x-1)^2}\, dx.$

Proponemos

$$\frac{x}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} = \frac{A(x-1)+B}{(x-1)^2} = \frac{Ax+(B-A)}{(x-1)^2},$$

de donde

$$\begin{aligned} A &= 1 \\ B-A &= 0. \end{aligned}$$

Al resolver el sistema de ecuaciones se obtiene

$$A = B = 1,$$

por lo que

$$\frac{x}{(x-1)^2} = \frac{1}{x-1} + \frac{1}{(x-1)^2}.$$

Por lo tanto,

$$\begin{aligned} \int \frac{x}{(x-1)^2}\, dx &= \int \left(\frac{1}{x-1} + \frac{1}{(x-1)^2}\right) dx = \int \frac{dx}{x-1} + \int \frac{dx}{(x-1)^2} \\ & \ln|x-1| - \frac{1}{x-1} + C. \end{aligned}$$

(d) $\int \frac{1}{x^2-x^3}\, dx.$

Proponemos

$$\begin{aligned} \frac{1}{x^2-x^3} &= \frac{1}{x^2(1-x)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{1-x} = \frac{Ax(1-x)+B(1-x)+Cx^2}{x^2(1-x)} \\ &= \frac{(C-A)x^2+(A-B)x+B}{x^2(1-x)}, \end{aligned}$$

de donde

$$\begin{aligned} C-A &= 0 \\ A-B &= 0 \\ B &= 1. \end{aligned}$$

Al resolver el sistema de ecuaciones se obtiene

$$A = B = C = 1,$$

por lo que

$$\frac{1}{x^2-x^3} = \frac{1}{x} + \frac{1}{x^2} + \frac{1}{1-x}.$$

Por lo tanto,

$$\begin{aligned}\int \frac{1}{x^2-x^3}\,dx &= \int \left(\frac{1}{x}+\frac{1}{x^2}+\frac{1}{1-x}\right)\,dx \\ &= \int \frac{dx}{x}+\int \frac{dx}{x^2}+\int \frac{dx}{1-x} \\ &= \ln|x|-\frac{1}{x}-\ln|1-x|+C.\end{aligned}$$

(e) $\displaystyle\int \frac{x^2}{x^2-1}\,dx.$

Primero reducimos la fracción impropia en el integrando, quedando

$$\frac{x^2}{x^2-1}=1+\frac{1}{x^2-1}.$$

De este modo,

$$\int \frac{x^2}{x^2-1}\,dx=\int \left(1+\frac{1}{x^2-1}\right)dx.$$

Proponemos

$$\begin{aligned}\frac{1}{x^2-1} &= \frac{A}{x+1}+\frac{B}{x-1}=\frac{A\,(x-1)+B\,(x+1)}{(x+1)\,(x-1)} \\ &= \frac{(A+B)\,x+(B-A)}{(x+1)\,(x-1)},\end{aligned}$$

de donde

$$\begin{aligned}A+B &= 0 \\ B-A &= 1.\end{aligned}$$

Al resolver el sistema de ecuaciones se obtiene

$$A=-1/2 \quad \text{y} \quad B=1/2,$$

de modo que

$$\frac{1}{x^2-1}=-\frac{1/2}{x+1}+\frac{1/2}{x-1}.$$

Por lo tanto,

$$\begin{aligned}\int \frac{x^2}{x^2-1}\,dx &= \int \left(1+\frac{1}{x^2-1}\right)dx=\int \left(1-\frac{1/2}{x+1}+\frac{1/2}{x-1}\right)\,dx \\ &= \int dx-\frac{1}{2}\int \frac{dx}{x+1}+\frac{1}{2}\int \frac{dx}{x-1} \\ &= x-\frac{1}{2}\ln|x+1|+\frac{1}{2}\ln|x-1|+C.\end{aligned}$$

(f) $\displaystyle\int \frac{x^3 - x^2 + 1}{x^2 - x}\, dx.$

Primero reducimos la fracción impropia en el integrando, quedando

$$\frac{x^3 - x^2 + 1}{x^2 - x} = x + \frac{1}{x^2 - x}.$$

De este modo,

$$\int \frac{x^3 - x^2 + 1}{x^2 - x}\, dx = \int \left(x + \frac{1}{x^2 - x} \right) dx.$$

Proponemos

$$\begin{aligned} \frac{1}{x^2 - x} &= \frac{1}{x\,(x-1)} = \frac{A}{x} + \frac{B}{x-1} \\ &= \frac{A\,(x-1) + Bx}{x\,(x-1)} = \frac{(A+B)\,x - A}{x\,(x-1)}, \end{aligned}$$

de donde

$$\begin{aligned} A + B &= 0 \\ -A &= 1. \end{aligned}$$

Resolviendo el sistema de ecuaciones se obtiene

$$A = -1 \text{ y } B = 1,$$

de modo que

$$\frac{1}{x^2 - x} = -\frac{1}{x} + \frac{1}{x-1}.$$

Por lo tanto,

$$\begin{aligned} \int \frac{x^3 - x^2 + 1}{x^2 - x}\, dx &= \int \left(x + \frac{1}{x^2 - x} \right) dx = \int \left(x - \frac{1}{x} + \frac{1}{x-1} \right)\, dx \\ &= \int x\, dx - \int \frac{dx}{x} + \int \frac{dx}{x-1} \\ &= \frac{x^2}{2} - \ln|x| + \ln|x-1| + C. \end{aligned}$$

(g) $\displaystyle\int \frac{x^4 + x^2 - 1}{x^3 + x}\, dx.$

Primero reducimos la fracción impropia en el integrando, quedando

$$\frac{x^4 + x^2 - 1}{x^3 + x} = x - \frac{1}{x^3 + x} = x - \frac{1}{x\,(x^2+1)}.$$

De este modo,

$$\int \frac{x^4 + x^2 - 1}{x^3 + x}\, dx = \int \left(x - \frac{1}{x\,(x^2+1)} \right)\, dx.$$

Proponemos

$$\begin{aligned}\frac{1}{x\left(x^2+1\right)} &= \frac{A}{x}+\frac{Bx+C}{x^2+1}=\frac{A\left(x^2+1\right)+\left(Bx+C\right)x}{x\left(x^2+1\right)}\\ &= \frac{\left(A+B\right)x^2+Cx+A}{x\left(x^2+1\right)},\end{aligned}$$

de donde

$$\begin{aligned}A+B &= 0\\ C &= 0\\ A &= 1.\end{aligned}$$

Al resolver el sistema de ecuaciones se obtiene

$$A=1,\ B=-1 \text{ y } C=0,$$

por lo que

$$\frac{1}{x\left(x^2+1\right)}=\frac{1}{x}+\frac{-x}{x^2+1}.$$

Por lo tanto,

$$\begin{aligned}\int\frac{x^4+x^2-1}{x^3+x}\,dx &= \int\left(x-\frac{1}{x\left(x^2+1\right)}\right)dx=\int\left(x-\left(\frac{1}{x}+\frac{-x}{x^2+1}\right)\right)dx\\ &= \int xdx-\int\frac{dx}{x}+\int\frac{x}{x^2+1}dx\\ &= \frac{x^2}{2}-\ln|x|+\frac{1}{2}\ln\left(x^2+1\right)+C.\end{aligned}$$

6. (a) $\displaystyle\int\frac{\ln x}{x}\,dx.$

Se propone la sustitución $u=\ln x$. Se tiene

$$u=\ln x \quad\Longrightarrow\quad du=\frac{1}{x}\,dx.$$

Por lo tanto,

$$\int\frac{\ln x}{x}\,dx=\int u\,du=\frac{u^2}{2}+C=\frac{\left(\ln x\right)^2}{2}+C=\frac{1}{2}\ln^2 x+C.$$

Nota: también puede utilizarse una integración por partes.

(b) $\displaystyle\int\frac{\ln x}{x^2}\,dx.$

Se integra por partes. Para ello, proponemos

$$\begin{aligned}u &= \ln x \quad\Longrightarrow\quad du=\frac{1}{x}\,dx\\ dv &= \frac{1}{x^2}\,dx \quad\Longrightarrow\quad v=-\frac{1}{x}.\end{aligned}$$

Por lo tanto,

$$\begin{aligned}\int \frac{\ln x}{x^2}\,dx &= -\frac{1}{x}\ln x - \int\left(-\frac{1}{x}\right)\left(\frac{1}{x}\right)\,dx = -\frac{1}{x}\ln x + \int \frac{1}{x^2}\,dx \\ &= -\frac{1}{x}\ln x - \frac{1}{x} + C.\end{aligned}$$

(c) $\displaystyle\int \frac{sen\,x}{1+\cos^2 x}\,dx.$

Se propone la sustitución $u = sen\,x$. Se tiene

$$u = \cos x \quad \Longrightarrow \quad du = -sen\,x\,dx.$$

Por lo tanto,

$$\int \frac{sen\,x}{1+\cos^2 x}\,dx = -\int \frac{du}{1+u^2} = -\tan^{-1} u + C = -\tan^{-1}(\cos x) + C.$$

(d) $\displaystyle\int \frac{1}{1+\cos x}\,dx.$

Se utilizan procedimientos algebraicos e identidades trigonométricas:

$$\begin{aligned}\int \frac{1}{1+\cos x}\,dx &= \int \frac{1}{1+\cos x}\left(\frac{1-\cos x}{1-\cos x}\right)\,dx = \int \frac{1-\cos x}{1-\cos^2 x}\,dx = \int \frac{1-\cos x}{sen^2 x}\,dx \\ &= \int \left[\csc^2 x - \csc x \cot x\right]\,dx = -\cot x + \csc x + C.\end{aligned}$$

(e) $\displaystyle\int \ln\left(1+x^2\right)\,dx.$

Se integra por partes. Para ello, proponemos

$$\begin{aligned} u &= \ln\left(1+x^2\right) \quad \Longrightarrow \quad du = \frac{2x}{1+x^2}\,dx \\ dv &= dx \quad \Longrightarrow \quad v = x.\end{aligned}$$

Por lo tanto,

$$\begin{aligned}\int \ln\left(1+x^2\right)\,dx &= x\ln\left(1+x^2\right) - 2\int \frac{x^2}{1+x^2}\,dx \\ &= x\ln\left(1+x^2\right) - 2\int\left(1-\frac{1}{1+x^2}\right)dx \\ &= x\ln\left(1+x^2\right) - 2x + 2\tan^{-1}(x) + C.\end{aligned}$$

(f) $\displaystyle\int \frac{x\,dx}{x^4+1}.$

Se integra por sustitución. Para ello, primero nota que

$$\int \frac{x\,dx}{x^4+1} = \int \frac{x\,dx}{\left(x^2\right)^2+1}.$$

Sea $u = x^2$. Se tiene

$$u = x^2 \quad \Longrightarrow \quad du = 2x\, dx.$$

Por lo tanto,

$$\int \frac{x\, dx}{x^4+1} = \frac{1}{2}\int \frac{2x\, dx}{(x^2)^2+1} = \frac{1}{2}\int \frac{du}{u^2+1} = \frac{1}{2}\tan^{-1}(u) + C = \frac{1}{2}\tan^{-1}\left(x^2\right) + C.$$

(g) $\displaystyle\int \frac{5x-3}{x^2-2x-3}\, dx.$

Se integra por fracciones parciales. Para ello, proponemos

$$\begin{aligned} \frac{5x-3}{x^2-2x-3} &= \frac{5x-3}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3} \\ &= \frac{A(x-3)+B(x+1)}{(x+1)(x-3)} = \frac{(A+B)x+(-3A+B)}{(x+1)(x-3)}, \end{aligned}$$

de donde

$$\begin{aligned} A+B &= 5 \\ -3A+B &= -3. \end{aligned}$$

Al resolver el sistema de ecuaciones se obtiene

$$A = 2 \text{ y } B = 3,$$

por lo que

$$\frac{5x-3}{x^2-2x-3} = \frac{2}{x+1} + \frac{3}{x-3}.$$

Por lo tanto,

$$\begin{aligned} \int \frac{5x-3}{x^2-2x-3}\, dx &= \int \left(\frac{2}{x+1} + \frac{3}{x-3}\right) dx = 2\int \frac{dx}{x+1} + 3\int \frac{dx}{x-3} \\ &= 2\ln|x+1| + 3\ln|x-3| + C. \end{aligned}$$

(h) $\displaystyle\int \frac{dx}{x-\sqrt{x}}$

Se integra por sustitución. Para ello, nota que

$$\int \frac{dx}{x-\sqrt{x}} = \int \frac{dx}{\sqrt{x}\left(\sqrt{x}-1\right)}.$$

Sea $u = \sqrt{x} - 1$. Se tiene

$$u = \sqrt{x} - 1 \quad \Longrightarrow \quad du = \frac{1}{2\sqrt{x}}\, dx.$$

Por lo tanto,

$$\int \frac{dx}{x-\sqrt{x}} = 2\int \frac{1}{\sqrt{x}-1}\left(\frac{1}{2\sqrt{x}}\right) dx = 2\int \frac{du}{u} = 2\ln|u| + C = 2\ln\left|\sqrt{x}-1\right| + C.$$

(i) $\displaystyle\int sen^{-1}(x)\ dx.$

Se integra por partes. Para ello, proponemos

$$\begin{aligned} u &= sen^{-1}(x) \quad \Longrightarrow \quad du = \frac{dx}{\sqrt{1-x^2}} \\ dv &= dx \quad \Longrightarrow \quad v = x. \end{aligned}$$

Por lo tanto,

$$\begin{aligned} \int sen^{-1}(x)\ dx &= x\ sen^{-1}(x) - \int \frac{x}{\sqrt{1-x^2}}\ dx \\ &= x\ sen^{-1}(x) + \sqrt{1-x^2} + C. \end{aligned}$$

CÁLCULO III
TAREA 7 - SOLUCIONES
FORMAS INDETERMINADAS. REGLA DE L´HOPITAL
(Tema 2.1)

1. (a)

$$\lim_{x\to 2}\frac{x-2}{x^2-4}\stackrel{L}{=}\lim_{x\to 2}\frac{1}{2x}=\frac{1}{4}.$$

(b) Aquí no se aplica la regla de L´Hopital, ya que el límite no es del tipo $\frac{0}{0}$, sino $\frac{0}{6}$. Se tiene

$$\lim_{x\to 3}\frac{x-3}{x^2-3}=\frac{3-3}{9-3}=\frac{0}{6}=0.$$

(c)

$$\lim_{x\to 0}\frac{3^x-1}{2^x-1}\stackrel{L}{=}\lim_{x\to 0}\frac{3^x\ln 3}{2^x\ln 2}=\frac{\ln 3}{\ln 2}.$$

(d)

$$\lim_{x\to 0}\frac{\ln(\cos 3x)}{2x^2}\stackrel{L}{=}\lim_{x\to 0}\frac{\frac{-3sen(3x)}{\cos(3x)}}{4x}=-\frac{3}{4}\lim_{x\to 0}\frac{\tan(3x)}{x}\stackrel{L}{=}-\frac{3}{4}\lim_{x\to 0}\frac{3\sec^2(3x)}{1}=-\frac{9}{4}.$$

(e)

$$\lim_{t\to 0}\frac{t\,sen\,t}{1-\cos t}\stackrel{L}{=}\lim_{t\to 0}\frac{t\cos t+sen\,t}{sen\,t}\stackrel{L}{=}\lim_{t\to 0}\frac{-t\,sen\,t+\cos t+\cos t}{\cos t}=2.$$

(f)

$$\lim_{x\to 0}\left(\frac{3^x}{x}-\frac{1}{x}\right)=\lim_{x\to 0}\frac{3^x-1}{x}\stackrel{L}{=}\lim_{x\to 0}\frac{3^x\ln 3}{1}=\ln 3.$$

(g) Aquí no se aplica la regla de L´Hopital, ya que el límite no es del tipo $\frac{0}{0}$, sino $\frac{-a}{0}$. Se tiene

$$\lim_{x\to 0^+}\frac{\sqrt{ax+x^2}-a}{x}=-\infty \quad \text{y} \quad \lim_{x\to 0^-}\frac{\sqrt{ax+x^2}-a}{x}=\infty.$$

(h)

$$\lim_{x\to 0}\frac{\int_0^x\sqrt{1+sen\,t}\,dt}{x}\stackrel{L}{=}\lim_{x\to 0}\frac{\frac{d}{dx}\int_0^x\sqrt{1+sen\,t}\,dt}{1}=\lim_{x\to 0}\frac{\sqrt{1+sen\,x}}{1}=1.$$

(i)

$$\begin{aligned}\lim_{x\to 1}\frac{\int_2^{2x}e^{4x-t^2}\,dt}{x-1}\stackrel{L}{=}\lim_{x\to 1}\frac{\frac{d}{dx}\left[e^{4x}\int_2^{2x}e^{-t^2}\,dt\right]}{1} &= \lim_{x\to 1}\frac{4e^{4x}\int_2^{2x}e^{-t^2}\,dt+e^{4x}\,e^{-(2x)^2}\frac{d(2x)}{dx}}{1}\\ &= \lim_{x\to 1}\left[4e^{4x}\int_2^{2x}e^{-t^2}\,dt+2e^{4x-4x^2}\right]\\ &= 4e^4\int_2^2 e^{-t^2}\,dt+2e^{4-4}=0+2=2.\end{aligned}$$

(j)

$$\lim_{x\to\infty} \frac{x-8x^2}{12x^2+5} \overset{L}{=} \lim_{x\to\infty} \frac{1-16x}{24x} \overset{L}{=} \lim_{x\to\infty} \left(-\frac{16}{24}\right) = -\frac{2}{3}.$$

(k)

$$\lim_{x\to\infty} \frac{x^{100}}{e^x} \overset{L}{=} \lim_{x\to\infty} \frac{100x^{99}}{e^x} \overset{L}{=} \lim_{x\to\infty} \frac{(100)(99)x^{98}}{e^x} \overset{L}{=} \ldots \overset{L}{=} (100)(99)\ldots(1) \underbrace{\left(\lim_{x\to\infty} \frac{1}{e^x}\right)}_{0} = 0.$$

(l)

$$\lim_{x\to\infty} \frac{x^{1/10}}{\ln x} \overset{L}{=} \lim_{x\to\infty} \frac{(1/10)\,x^{-9/10}}{1/x} = \frac{1}{10} \lim_{x\to\infty} \frac{x}{x^{9/10}} = \frac{1}{10} \lim_{x\to\infty} x^{1/10} = \infty.$$

(m)

$$\lim_{x\to\infty} \frac{\ln(x^2+2x)}{\ln x} \overset{L}{=} \lim_{x\to\infty} \frac{\frac{2x+2}{x^2+2x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{x(2x+2)}{x^2+2x} = \lim_{x\to\infty} \frac{2x+2}{x+2} \overset{L}{=} \lim_{x\to\infty} \frac{2}{1} = 2.$$

(n)

$$\lim_{x\to\infty} \frac{x\ln x}{\ln(\ln x)} \overset{L}{=} \lim_{x\to\infty} \frac{x\frac{1}{x}+\ln x}{\frac{1/x}{\ln x}} = \lim_{x\to\infty} \left[(x\ln x)(1+\ln x)\right] = \infty.$$

(o)

$$\lim_{x\to\infty} \left(x^2 e^{-x}\right) = \lim_{x\to\infty} \frac{x^2}{e^x} \overset{L}{=} \lim_{x\to\infty} \frac{2x}{e^x} \overset{L}{=} \lim_{x\to\infty} \frac{2}{e^x} = 0.$$

(p)

$$\lim_{x\to 0^+} \left(xe^{1/x}\right) = \lim_{x\to 0^+} \frac{e^{1/x}}{1/x} \overset{L}{=} \lim_{x\to 0^+} \frac{e^{1/x}(-1/x^2)}{-1/x^2} = \lim_{x\to 0^+} e^{1/x} = \infty.$$

(q)

$$\lim_{x\to\infty} \left(xe^{-1/x}\right) = \underbrace{\lim_{x\to\infty} \frac{e^{-1/x}}{1/x}}_{\text{no se usa L'Hopital}} = \infty.$$

(r)

$$\lim_{x\to\infty} \left[x\ln(1+3/x)\right] = \lim_{x\to\infty} \frac{\ln(1+3/x)}{1/x} \overset{L}{=} \lim_{x\to\infty} \frac{\frac{-3/x^2}{1+3/x}}{-1/x^2} = \lim_{x\to\infty} \frac{3}{\frac{3}{x}+1} = 3.$$

(s)

$$\lim_{x\to\infty} \left[e^x \ln\left(1+e^{-x}\right)\right] = \lim_{x\to\infty} \left[\frac{\ln(1+e^{-x})}{e^{-x}}\right] \overset{L}{=} \lim_{x\to\infty} \frac{\frac{-e^{-x}}{1+e^{-x}}}{-e^{-x}} = \lim_{x\to\infty} \frac{1}{1+e^{-x}} = 1.$$

(t)

$$\lim_{x\to 0^+} \left(x^2 \ln x\right) = \lim_{x\to 0^+} \frac{\ln x}{1/x^2} \overset{L}{=} \lim_{x\to 0^+} \frac{1/x}{(-2/x^3)} = -\frac{1}{2} \lim_{x\to 0^+} x^2 = 0.$$

(u) Aquí no se aplica la regla de L´Hopital, ya que el límite no es del tipo $0 \cdot \infty$, sino $\infty \cdot \infty$. Se tiene

$$\lim_{x\to\infty} \left(x^2 \ln x\right) = \infty.$$

(v)

$$\begin{aligned}\lim_{x\to 0^+} [\ln x - \ln(sen\, x)] &= \lim_{x\to 0^+} \ln\left(\frac{x}{sen\, x}\right) \\ &= \ln\left(\lim_{x\to 0^+} \frac{x}{sen\, x}\right) \overset{L}{=} \ln\left(\lim_{x\to 0^+} \frac{1}{\cos x}\right) = \ln 1 = 0.\end{aligned}$$

(w)

$$\begin{aligned}\lim_{x\to\infty} [\ln(2x) - \ln(x+1)] &= \lim_{x\to\infty} \ln\left(\frac{2x}{x+1}\right) \\ &= \ln\left(\lim_{x\to\infty} \frac{2x}{x+1}\right) \overset{L}{=} \ln\left(\lim_{x\to\infty} \frac{2}{1}\right) = \ln 2.\end{aligned}$$

(x) Aquí no se aplica la regla de L´Hopital, ya que el límite no es del tipo $\infty - \infty$, sino $\infty - 0$. Se tiene

$$\lim_{x\to 0^+} [\ln(2x) - \ln(x+1)] = \lim_{x\to 0^+} \ln\left(\frac{2x}{x+1}\right) = \ln\left(\lim_{x\to 0^+} \frac{2x}{x+1}\right) = \ln(0) = -\infty.$$

2. Utilizando la regla de L'Hopital:

$$\lim_{x\to 0^+} \frac{\cot x}{\csc x} \overset{L}{=} \lim_{x\to 0^+} \frac{-\csc^2 x}{-\csc x \cot x} = \lim_{x\to 0^+} \frac{\csc x}{\cot x} \overset{L}{=} \lim_{x\to 0^+} \frac{-\csc x \cot x}{-\csc^2 x} = \underbrace{\lim_{x\to 0^+} \frac{\cot x}{\csc x}}_{\text{límite original}}.$$

Utilizando identidades trigonométricas:

$$\lim_{x\to 0^+} \frac{\cot x}{\csc x} = \lim_{x\to 0^+} \frac{(\cos x/senx)}{(1/senx)} = \lim_{x\to 0^+} (\cos x) = 1.$$

3. (a)

$$\lim_{x\to 0^+} x^{2/x} = \lim_{x\to 0^+} e^{\ln x^{2/x}} = e^{\lim\limits_{x\to 0^+} \ln x^{2/x}} = e^{\lim\limits_{x\to 0^+} \left[\frac{2}{x}\ln x\right]} = e^{-\infty} = 0.$$

(b)

$$\begin{aligned}\lim_{x\to 1^+} x^{1/(x-1)} &= \lim_{x\to 1^+} e^{\ln x^{1/(x-1)}} = e^{\lim\limits_{x\to 1^+} \ln x^{1/(x-1)}} = e^{\lim\limits_{x\to 1^+} \left[\frac{1}{x-1}\ln x\right]} \\ &= e^{\lim\limits_{x\to 1^+} \frac{\ln x}{x-1}} \overset{L}{=} e^{\lim\limits_{x\to 1^+} \frac{1/x}{1}} = e.\end{aligned}$$

(c)

$$\begin{aligned}\lim_{x\to\infty} (\ln x)^{1/x} &= \lim_{x\to\infty} e^{\ln(\ln x)^{1/x}} = e^{\lim\limits_{x\to\infty} \ln(\ln x)^{1/x}} = e^{\lim\limits_{x\to\infty} \left[\frac{1}{x}\ln(\ln x)\right]} \\ &= e^{\lim\limits_{x\to\infty} \frac{\ln(\ln x)}{x}} \overset{L}{=} e^{\lim\limits_{x\to\infty} \frac{\left(\frac{1/x}{\ln x}\right)}{1}} = e^{\lim\limits_{x\to\infty} \frac{1}{x\ln x}} = e^0 = 1.\end{aligned}$$

(d)

$$\lim_{x\to\infty} x^{1/\ln x} = \lim_{x\to\infty} e^{\ln x^{1/\ln x}} = e^{\lim\limits_{x\to\infty} \ln x^{1/\ln x}} = e^{\lim\limits_{x\to\infty}\left[\frac{1}{\ln x}\ln x\right]} = \underbrace{\lim_{x\to\infty} e^{1}}_{\text{no se usa L'Hopital}} = e.$$

(e)

$$\begin{aligned}\lim_{x\to 0^+} x^x &= \lim_{x\to 0^+} e^{\ln x^x} = e^{\lim\limits_{x\to 0^+} \ln x^x} = e^{\lim\limits_{x\to 0^+} [x\ln x]} \\ &= e^{\lim\limits_{x\to 0^+} \frac{\ln x}{(1/x)}} \stackrel{L}{=} e^{\lim\limits_{x\to 0^+} \frac{(1/x)}{(-1/x^2)}} = e^{-\lim\limits_{x\to 0^+} x} = e^0 = 1.\end{aligned}$$

(f)

$$\begin{aligned}\lim_{x\to 0^+} (e^x+x)^{1/x} &= \lim_{x\to 0^+} e^{\ln(e^x+x)^{1/x}} = e^{\lim\limits_{x\to 0^+} \ln(e^x+x)^{1/x}} = e^{\lim\limits_{x\to 0^+}\left[\frac{1}{x}\ln(e^x+x)\right]} \\ &= e^{\lim\limits_{x\to 0^+} \frac{\ln(e^x+x)}{x}} \stackrel{L}{=} e^{\lim\limits_{x\to 0^+} \frac{\frac{e^x+1}{e^x+x}}{1}} = e^{\lim\limits_{x\to 0^+} \frac{e^x+1}{e^x+x}} = e^2.\end{aligned}$$

(g)

$$\begin{aligned}\lim_{x\to 0^+} (1+2x)^{1/(3x)} &= \lim_{x\to 0^+} e^{\ln(1+2x)^{1/3x}} = e^{\lim\limits_{x\to 0^+} \ln(1+2x)^{1/3x}} = e^{\lim\limits_{x\to 0^+}\left[\frac{1}{3x}\ln(1+2x)\right]} \\ &= e^{\lim\limits_{x\to 0^+} \frac{\ln(1+2x)}{3x}} \stackrel{L}{=} e^{\lim\limits_{x\to 0^+} \frac{\frac{2}{1+2x}}{3}} = e^{2/3}.\end{aligned}$$

(h)

$$\begin{aligned}\lim_{x\to\infty} (1+2x)^{1/(2\ln x)} &= \lim_{x\to\infty} e^{\ln(1+2x)^{1/(2\ln x)}} = e^{\lim\limits_{x\to\infty} \ln(1+2x)^{1/(2\ln x)}} = e^{\lim\limits_{x\to\infty}\left[\frac{1}{2\ln x}\ln(1+2x)\right]} \\ &= e^{\lim\limits_{x\to\infty} \frac{\ln(1+2x)}{2\ln x}} \stackrel{L}{=} e^{\lim\limits_{x\to\infty} \frac{\frac{2}{1+2x}}{2/x}} = e^{\lim\limits_{x\to\infty} \frac{x}{1+2x}} \stackrel{L}{=} e^{\lim\limits_{x\to\infty} \frac{1}{2}} = \sqrt{e}.\end{aligned}$$

(i)

$$\begin{aligned}\lim_{x\to 0^+} (1+2x)^{1/(2\ln x)} &= \lim_{x\to 0^+} e^{(1+2x)^{1/(2\ln x)}} = e^{\lim\limits_{x\to 0^+} (1+2x)^{1/(2\ln x)}} \\ &= e^{\lim\limits_{x\to 0^+}\left[\frac{1}{2\ln x}\ln(1+2x)\right]} = \underbrace{e^{\lim\limits_{x\to 0^+} \frac{\ln(1+2x)}{2\ln x}}}_{\text{no se usa L'Hopital}} = e^{\frac{0}{-\infty}} = 1.\end{aligned}$$

(j)

$$\begin{aligned}\lim_{x\to\infty} (2^x+1)^{1/x} &= \lim_{x\to\infty} e^{\ln(2^x+1)^{1/x}} \\ &= e^{\lim\limits_{x\to\infty} \ln(2^x+1)^{1/x}} = e^{\lim\limits_{x\to\infty}\left[\frac{1}{x}\ln(2^x+1^x)\right]} \\ &= e^{\lim\limits_{x\to\infty} \frac{\ln(2^x+1)}{x}} \stackrel{L}{=} e^{\lim\limits_{x\to\infty} \frac{\frac{2^x\ln(2)}{2^x+1}}{1}} \\ &= e^{\lim\limits_{x\to\infty} \frac{2^x\ln(2)}{2^x+1}} \stackrel{L}{=} e^{\lim\limits_{x\to\infty} \frac{2^x\ln(2)\ln(2)}{2^x\ln(2)}} \\ &= e^{\lim\limits_{x\to\infty} \ln(2)} = e^{\ln(2)} = 2.\end{aligned}$$

4.

$$\lim_{a\to 1}\frac{x^{1-a}-1}{1-a}\overset{L}{=}\lim_{a\to 1}\frac{\frac{d}{da}\left(x^{1-a}-1\right)}{\frac{d}{da}\left(1-a\right)}=\lim_{a\to 1}\frac{x^{1-a}\ln x\left(-1\right)}{\left(-1\right)}=\lim_{a\to 1}\left(x^{1-a}\ln x\right)=\ln x.$$

5.

$$\begin{aligned}\lim_{k\to\infty}A_0\left(1+r/k\right)^{kt} &= A_0\lim_{k\to\infty}\left[\left(1+r/k\right)^k\right]^t=A_0\left[\lim_{k\to\infty}\left(1+r/k\right)^k\right]^t\\ &= A_0\left[\lim_{k\to\infty}e^{\ln(1+r/k)^k}\right]^t=A_0\left[e^{\lim\limits_{k\to\infty}\ln(1+r/k)^k}\right]^t.\end{aligned}$$

Calculemos por separado $\lim\limits_{k\to\infty}\ln\left(1+r/k\right)^k$:

$$\begin{aligned}\lim_{k\to\infty}\ln\left(1+r/k\right)^k &= \lim_{k\to\infty}\left[k\ln\left(1+r/k\right)\right]=\lim_{k\to\infty}\frac{\ln\left(1+r/k\right)}{1/k}\overset{L}{=}\lim_{k\to\infty}\frac{\frac{d}{dk}\ln\left(1+r/k\right)}{\frac{d}{dk}\left(1/k\right)}\\ &= \lim_{k\to\infty}\frac{\frac{\left(-r/k^2\right)}{1+(r/k)}}{\left(-1/k^2\right)}=\lim_{k\to\infty}\frac{r}{1+r/k}=r.\end{aligned}$$

Por lo tanto,

$$\lim_{k\to\infty}A_0\left(1+r/k\right)^{kt}=A_0\left[e^{\lim\limits_{k\to\infty}\ln(1+r/k)^k}\right]^t=A_0\left(e^r\right)^t=A_0e^{rt}.$$

6.

$$\begin{aligned}\lim_{t\to 0^+}\left(\sum_{i=1}^n c_ix_i^t\right)^{1/t} &= \lim_{t\to 0^+}\left(c_1x_1^t+\cdots+c_nx_n^t\right)^{1/t}=\lim_{t\to 0^+}e^{\ln\left(c_1x_1^t+\cdots+c_nx_n^t\right)^{1/t}}\\ &= e^{\lim\limits_{t\to 0^+}\ln\left(c_1x_1^t+\cdots+c_nx_n^t\right)^{1/t}}=e^{\lim\limits_{t\to 0^+}\frac{\ln\left(c_1x_1^t+\cdots+c_nx_n^t\right)}{t}}.\end{aligned}$$

Nota que $\lim\limits_{t\to 0^+}\frac{\ln\left(c_1x_1^t+\cdots+c_nx_n^t\right)}{t}$ es del tipo $\frac{0}{0}$, ya que $c_1+...+c_n=1$. Usando la regla de L'Hopital,

$$\begin{aligned}\lim_{t\to 0^+}\frac{\ln\left(c_1x_1^t+\cdots+c_nx_n^t\right)}{t} &= \lim_{t\to 0^+}\frac{\frac{c_1x_1^t\ln x_1+...+c_nx_n^t\ln x_n}{c_1x_1^t+...+c_nx_n^t}}{1}\\ &= \frac{c_1\ln x_1+...+c_n\ln x_n}{c_1+...+c_n}\\ &= \frac{c_1\ln x_1+...+c_n\ln x_n}{1}\\ &= \ln x_1^{c_1}+...+\ln x_n^{c_n}\\ &= \ln\left(x_1^{c_1}\cdot\cdots\cdot x_n^{c_n}\right).\end{aligned}$$

Por lo tanto,

$$\lim_{t\to 0^+}\left(\sum_{i=1}^n c_ix_i^t\right)^{1/t}=e^{\lim\limits_{t\to 0^+}\frac{\ln\left(c_1x_1^t+\cdots+c_nx_n^t\right)}{t}}=e^{\ln\left(x_1^{c_1}\cdot\cdots\cdot x_n^{c_n}\right)}=x_1^{c_1}\cdot\cdots\cdot x_n^{c_n}=\prod_{i=1}^n x_i^{c_i}.$$

7. Sea $f(\sigma, v) = \dfrac{\left[\left(\gamma h^{1-\frac{1}{v}} + c^{1-\frac{1}{v}}\right)^{\frac{v}{v-1}}\right]^{1-1/\sigma}}{1 - 1/\sigma}$.

(a) Por simplicidad, definimos $x(v) = \left(\gamma h^{1-\frac{1}{v}} + c^{1-\frac{1}{v}}\right)^{\frac{v}{v-1}}$, con $x > 0$. Así,

$$f = \frac{x^{1-1/\sigma}}{1 - 1/\sigma}.$$

La transformación monotónica $y(\sigma, v) = f(\sigma, v) - \dfrac{1}{1 - 1/\sigma}$ se convierte en

$$y = f - \frac{1}{1 - 1/\sigma} = \frac{x^{1-1/\sigma}}{1 - 1/\sigma} - \frac{1}{1 - 1/\sigma} = \frac{x^{1-1/\sigma} - 1}{1 - 1/\sigma}.$$

De esta manera,

$$\begin{aligned}
\lim_{\sigma \to 1} y &= \lim_{\sigma \to 1} \frac{x^{1-1/\sigma} - 1}{1 - 1/\sigma} \stackrel{L}{=} \lim_{\sigma \to 1} \frac{\frac{d}{d\sigma}\left(x^{1-1/\sigma} - 1\right)}{\frac{d}{d\sigma}\left(1 - 1/\sigma\right)} = \lim_{\sigma \to 1} \frac{x^{1-1/\sigma}(\ln x)(1/\sigma^2)}{(1/\sigma^2)} \\
&= \lim_{\sigma \to 1} \left(x^{1-1/\sigma} \ln x\right) = \ln x.
\end{aligned}$$

Nota que este resultado coincide con el del ejercicio 4. Por último, sustituimos la función $x(v)$, de donde

$$\lim_{\sigma \to 1} y(\sigma, v) = \ln \left(\gamma h^{1-\frac{1}{v}} + c^{1-\frac{1}{v}}\right)^{\frac{v}{v-1}}.$$

(b) La transformación monotónica $y(\sigma, v) = \left(\dfrac{1}{1+\gamma}\right)^{\left(1-\frac{1}{\sigma}\right)\left(\frac{v}{v-1}\right)} f(\sigma, v)$ aquí es

$$\begin{aligned}
y &= \left(\frac{1}{1+\gamma}\right)^{\left(1-\frac{1}{\sigma}\right)\left(\frac{v}{v-1}\right)} f = \left(\frac{1}{1+\gamma}\right)^{\left(1-\frac{1}{\sigma}\right)\left(\frac{v}{v-1}\right)} \frac{\left[\left(\gamma h^{1-\frac{1}{v}} + c^{1-\frac{1}{v}}\right)^{\frac{v}{v-1}}\right]^{1-1/\sigma}}{1 - 1/\sigma} \\
&= \frac{1}{1 - 1/\sigma} \left[\left[\left(\frac{1}{1+\gamma}\right)\left(\gamma h^{1-\frac{1}{v}} + c^{1-\frac{1}{v}}\right)\right]^{\frac{v}{v-1}}\right]^{1-1/\sigma} \\
&= \frac{1}{1 - 1/\sigma} \left[\left(\frac{\gamma}{1+\gamma} h^{1-\frac{1}{v}} + \frac{1}{1+\gamma} c^{1-\frac{1}{v}}\right)^{\frac{v}{v-1}}\right]^{1-1/\sigma} = \frac{z^{1-1/\sigma}}{1 - 1/\sigma},
\end{aligned}$$

donde hemos definido, por simplicidad, $z(v) = \left(\frac{\gamma}{1+\gamma} h^{1-\frac{1}{v}} + \frac{1}{1+\gamma} c^{1-\frac{1}{v}}\right)^{\frac{v}{v-1}}$, con $z > 0$. De esta manera,

$$\lim_{v \to 1} y = \lim_{v \to 1} \frac{z^{1-1/\sigma}}{1 - 1/\sigma} = \frac{1}{1 - 1/\sigma} \left(\lim_{v \to 1} z\right)^{1-1/\sigma}.$$

Calculemos por separado $\lim_{v\to 1} z$:

$$\lim_{v\to 1} z(v) = \lim_{v\to 1}\left(\frac{\gamma}{1+\gamma}h^{1-\frac{1}{v}} + \frac{1}{1+\gamma}c^{1-\frac{1}{v}}\right)^{\frac{v}{v-1}} = e^{\lim_{v\to 1}\ln\left(\frac{\gamma}{1+\gamma}h^{1-\frac{1}{v}}+\frac{1}{1+\gamma}c^{1-\frac{1}{v}}\right)^{\frac{v}{v-1}}}.$$

Usando la regla de L'Hopital en el exponente, se tiene

$$\begin{aligned}
\lim_{v\to 1}\ \ln\left(\frac{\gamma}{1+\gamma}h^{1-\frac{1}{v}} + \frac{1}{1+\gamma}c^{1-\frac{1}{v}}\right)^{\frac{v}{v-1}} &= \lim_{v\to 1}\frac{\ln\left(\frac{\gamma}{1+\gamma}h^{1-\frac{1}{v}} + \frac{1}{1+\gamma}c^{1-\frac{1}{v}}\right)}{\left(\frac{v-1}{v}\right)} \\
&= \lim_{v\to 1}\frac{\ln\left(\frac{\gamma}{1+\gamma}h^{1-\frac{1}{v}} + \frac{1}{1+\gamma}c^{1-\frac{1}{v}}\right)}{1-1/v} \\
&= \lim_{v\to 1}\frac{\dfrac{\frac{\gamma}{1+\gamma}h^{1-\frac{1}{v}}(\ln h)\left(1/v^2\right)+\frac{1}{1+\gamma}c^{1-\frac{1}{v}}(\ln c)\left(1/v^2\right)}{\frac{\gamma}{1+\gamma}h^{1-\frac{1}{v}}+\frac{1}{1+\gamma}c^{1-\frac{1}{v}}}}{1/v^2} \\
&= \frac{\frac{\gamma}{1+\gamma}\ln h + \frac{1}{1+\gamma}\ln c}{\frac{\gamma}{1+\gamma} + \frac{1}{1+\gamma}} = \frac{\frac{\gamma}{1+\gamma}\ln h + \frac{1}{1+\gamma}\ln c}{1} \\
&= \ln\left(h^{\frac{\gamma}{1+\gamma}}c^{\frac{1}{1+\gamma}}\right).
\end{aligned}$$

$$\therefore\quad \lim_{v\to 1} z(v) = e^{\ln\left(h^{\frac{\gamma}{1+\gamma}}c^{\frac{1}{1+\gamma}}\right)} = h^{\frac{\gamma}{1+\gamma}}c^{\frac{1}{1+\gamma}}.$$

$$\therefore\quad \lim_{v\to 1} y(\sigma, v) = \frac{1}{1-1/\sigma}\left(h^{\frac{\gamma}{1+\gamma}}c^{\frac{1}{1+\gamma}}\right)^{1-1/\sigma}.$$

CÁLCULO III
TAREA 8 - SOLUCIONES
INTEGRALES IMPROPIAS
(Tema 2.2)

1. (a)

$$\int_{-\infty}^{-2} \frac{dx}{x^5} = \lim_{a\to-\infty} \int_a^{-2} x^{-5}\,dx = \lim_{a\to-\infty} \left[\frac{x^{-4}}{-4}\right]_a^{-2} = \lim_{a\to-\infty} \left[-\frac{1}{4(-2)^4} + \frac{1}{4a^4}\right]$$
$$= -\frac{1}{4(-2)^4} + \underbrace{\left(\lim_{a\to-\infty} \frac{1}{4a^4}\right)}_{0} = -\frac{1}{64}.$$

(b)

$$\int_2^{\infty} \frac{dt}{\sqrt{t-1}} = \lim_{b\to\infty} \int_2^{b} \frac{dt}{\sqrt{t-1}} = 2\lim_{b\to\infty}\left[\sqrt{t-1}\right]_2^b = 2\lim_{b\to\infty}\left[\sqrt{b-1}-\sqrt{2-1}\right]$$
$$= 2\underbrace{\left(\lim_{b\to\infty}\sqrt{b-1}\right)}_{\infty} - 2\sqrt{2-1} = \infty \quad \therefore \ \text{diverge.}$$

(c)

$$\int_\beta^{\infty} \frac{\alpha\beta^\alpha}{y^{\alpha+1}}\,dy = \alpha\beta^\alpha \lim_{b\to\infty} \int_\beta^b y^{-\alpha-1}\,dy = \alpha\beta^\alpha \lim_{b\to\infty}\left[\frac{y^{-\alpha}}{-\alpha}\right]_\beta^b = -\beta^\alpha \lim_{b\to\infty}\left[\frac{1}{b^\alpha} - \frac{1}{\beta^\alpha}\right]$$
$$= -\beta^\alpha \underbrace{\left(\lim_{b\to\infty}\frac{1}{b^\alpha}\right)}_{0\ (\alpha>0)} + 1 = 1.$$

(d)

$$\int_0^{\infty} e^{-2x}\,dx = \lim_{b\to\infty}\int_0^b e^{-2x}\,dx = -\frac{1}{2}\lim_{b\to\infty}\left[e^{-2x}\right]_0^b = -\frac{1}{2}\lim_{b\to\infty}\left[e^{-2b}-1\right]$$
$$= -\frac{1}{2}\underbrace{\left(\lim_{b\to\infty}\frac{1}{e^{2b}}\right)}_{0} + \frac{1}{2} = \frac{1}{2}.$$

(e)

$$\int_{-\infty}^{0} xe^{x/2}\,dx = \lim_{a\to-\infty}\int_a^0 xe^{x/2}\,dx = \lim_{a\to-\infty}\left[2xe^{x/2}-4e^{x/2}\right]_a^0$$
$$= \lim_{a\to-\infty}\left[(0-4)-\left(2ae^{a/2}-4e^{a/2}\right)\right]$$
$$= -4 - 2\underbrace{\left(\lim_{a\to-\infty} ae^{a/2}\right)}_{0\ (\text{* L'Hopital})} + 4\underbrace{\left(\lim_{a\to-\infty} e^{a/2}\right)}_{0} = -4.$$

* L'Hopital: $\lim_{a\to-\infty}\left(ae^{a/2}\right) = \lim_{a\to-\infty}\frac{a}{e^{-a/2}} \overset{L}{=} \lim_{a\to-\infty}\frac{1}{-\frac{1}{2}e^{-a/2}} = -2\lim_{a\to-\infty} e^{a/2} = 0.$

(f)

$$\int_4^\infty xe^{-x^2}\,dx = \lim_{b\to\infty}\int_4^b xe^{-x^2}\,dx = -\frac{1}{2}\lim_{b\to\infty}\left[e^{-x^2}\right]_4^b = -\frac{1}{2}\lim_{b\to\infty}\left[e^{-b^2}-e^{-16}\right]$$

$$= -\frac{1}{2}\underbrace{\left(\lim_{b\to\infty}\frac{1}{e^{b^2}}\right)}_{0} + \frac{1}{2e^{16}} = \frac{1}{2e^{16}}.$$

(g)

$$\int_2^\infty \frac{dx}{x\ln x} = \lim_{b\to\infty}\int_2^b \frac{dx/x}{\ln x} = \lim_{b\to\infty}\left[\ln|\ln x|\right]_2^b = \lim_{b\to\infty}\left[\ln|\ln b| - \ln|\ln 2|\right]$$

$$= \ln\underbrace{\left(\lim_{b\to\infty}|\ln b|\right)}_{\infty} - \ln|\ln 2| = \infty \quad \therefore \quad \text{diverge.}$$

(h)

$$\int_0^\infty \frac{e^{-\theta}\,d\theta}{1+e^{-\theta}} = \lim_{b\to\infty}\int_0^b \frac{e^{-\theta}\,d\theta}{1+e^{-\theta}} = -\lim_{b\to\infty}\left[\ln\left|1+e^{-\theta}\right|\right]_0^b$$

$$= -\lim_{b\to\infty}\left[\ln\left(1+\frac{1}{e^b}\right) - \ln 2\right] = -\ln\left[\underbrace{\lim_{b\to\infty}\left(1+\frac{1}{e^b}\right)}_{1}\right] + \ln 2$$

$$= -\underbrace{\ln 1}_{0} + \ln 2 = \ln 2.$$

(i)

$$\int_1^\infty \frac{dx}{x(3x+1)} = \lim_{b\to\infty}\int_1^b \frac{dx}{x(3x+1)} = \lim_{b\to\infty}\int_1^b \underbrace{\left(\frac{1}{x}-\frac{3}{3x+1}\right)}_{\text{fracciones parciales}}dx$$

$$= \lim_{b\to\infty}\left[\ln|x| - \ln|3x+1|\right]_1^b$$

$$= \lim_{b\to\infty}\left[\ln|b| - \ln|3b+1| - \ln|1| + \ln|4|\right] \qquad (b>0)$$

$$= \lim_{b\to\infty}\ln\left(\frac{b}{3b+1}\right) + \ln 4 = \ln\underbrace{\left(\lim_{b\to\infty}\frac{b}{3b+1}\right)}_{1/3\ (\text{* L´Hopital})} + \ln 4 = \ln\left(\frac{4}{3}\right).$$

$$\text{* L'Hopital:}\quad \lim_{b\to\infty}\frac{b}{3b+1} = \lim_{b\to\infty}\frac{1}{3} = \frac{1}{3}.$$

(j)

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2} = \int_{-\infty}^{0} \frac{dx}{x^2+2x+2} + \int_{0}^{\infty} \frac{dx}{x^2+2x+2}$$
$$= \lim_{a\to-\infty} \int_a^0 \frac{dx}{(x+1)^2+1} + \lim_{b\to\infty} \int_0^b \frac{dx}{(x+1)^2+1}$$
$$= \lim_{a\to-\infty} \left[\tan^{-1}(x+1)\right]_a^0 + \lim_{b\to\infty} \left[\tan^{-1}(x+1)\right]_0^b$$
$$= \lim_{a\to-\infty} \left[\tan^{-1}(1) - \tan^{-1}(a+1)\right] + \lim_{b\to\infty} \left[\tan^{-1}(b+1) - \tan^{-1}(1)\right]$$
$$= \tan^{-1}(1) - \lim_{a\to-\infty} \tan^{-1}(a+1) + \lim_{b\to\infty} \tan^{-1}(b+1) - \tan^{-1}(1)$$
$$= -\underbrace{\left(\lim_{a\to-\infty} \tan^{-1}(a+1)\right)}_{-\pi/2} + \underbrace{\left(\lim_{b\to\infty} \tan^{-1}(b+1)\right)}_{\pi/2} = \pi.$$

2. (a)

$$\int_2^{10} \frac{dt}{\sqrt{2t-4}} = \lim_{a\to2^+} \int_a^{10} \frac{dt}{\sqrt{2t-4}} = \lim_{a\to2^+} \left[\sqrt{2t-4}\right]_a^{10} = \lim_{a\to2^+} \left[4-\sqrt{2a-4}\right]$$
$$= 4 - \underbrace{\left(\lim_{a\to2^+} \sqrt{2a-4}\right)}_{0} = 4.$$

(b) Método 1:

$$\int_{-3}^{-2} \frac{dt}{(6+3t)^2} = \lim_{b\to-2^-} \int_{-3}^{b} \frac{dt}{(6+3t)^2} = -\frac{1}{3} \lim_{b\to-2^-} \left[\frac{1}{6+3t}\right]_{-3}^{b}$$
$$= -\frac{1}{3} \lim_{b\to-2^-} \left[\frac{1}{6+3b} + \frac{1}{3}\right]$$
$$= -\frac{1}{3} \underbrace{\left(\lim_{b\to-2^-} \frac{1}{6+3b}\right)}_{-\infty} - \frac{1}{9} = \infty \quad \therefore \quad \text{diverge.}$$

Método 2 (con cambio de variable):

Sea $u = 6+3t$. Se tiene

$$u = 6+3t \implies du = 3\,dt,\ u(-3) = -3,\ u(-2) = 0.$$

Por lo tanto,

$$\int_{-3}^{-2} \frac{dt}{(6+3t)^2} = \frac{1}{3} \int_{-3}^{0} \frac{du}{u^2} = \frac{1}{3} \lim_{b\to0^-} \int_{-3}^{b} \frac{du}{u^2} = \frac{1}{3} \lim_{b\to0^-} \left[-\frac{1}{u}\right]_{-3}^{b}$$
$$= \frac{1}{3} \lim_{b\to0^-} \left[-\frac{1}{b} - \frac{1}{3}\right]$$
$$= -\frac{1}{3} \underbrace{\left(\lim_{b\to0^-} \frac{1}{b}\right)}_{-\infty} - \frac{1}{9} = \infty \quad \therefore \quad \text{diverge.}$$

(c)

$$\begin{aligned}\int_0^{1/\ln 2} \frac{e^{-1/x}}{x^2}\,dx &= \lim_{a\to 0^+}\int_a^{1/\ln 2} \frac{e^{-1/x}}{x^2}\,dx = \lim_{a\to 0^+}\left[e^{-1/x}\right]_a^{1/\ln 2} = \lim_{a\to 0^+}\left[e^{-\ln 2}-e^{-1/a}\right]\\ &= \frac{1}{e^{\ln 2}} - \underbrace{\left(\lim_{a\to 0^+} e^{-1/a}\right)}_{0} = \frac{1}{2}.\end{aligned}$$

(d)

$$\begin{aligned}\int_0^1 x\ln x\,dx &= \lim_{a\to 0^+}\int_a^1 x\ln x\,dx = \lim_{a\to 0^+}\left[\frac{x^2}{2}\ln x - \frac{x^2}{4}\right]_a^1\\ &= \lim_{a\to 0^+}\left[\left(\frac{1}{2}\underbrace{\ln 1}_{0} - \frac{1}{4}\right) - \left(\frac{a^2}{2}\ln a - \frac{a^2}{4}\right)\right]\\ &= -\frac{1}{4} - \frac{1}{2}\underbrace{\lim_{a\to 0^+}\left(a^2\ln a\right)}_{0\ (*\ \text{L\'Hopital})} + \frac{1}{4}\underbrace{\lim_{a\to 0^+} a^2}_{0} = -\frac{1}{4}.\end{aligned}$$

* L'Hopital: $\lim_{a\to 0^+}\left(a^2\ln a\right) = \lim_{a\to 0^+}\frac{\ln a}{1/a^2} \stackrel{L}{=} \lim_{a\to 0^+}\left(\frac{1/a}{-2/a^3}\right) = -\frac{1}{2}\lim_{a\to 0^+} a^2 = 0.$

(e)

$$\begin{aligned}\int_0^2 \frac{s+1}{\sqrt{4-s^2}}\,ds &= \lim_{b\to 2^-}\int_0^b \frac{s+1}{\sqrt{4-s^2}}\,ds = \lim_{b\to 2^-}\left[\int_0^b \frac{s}{\sqrt{4-s^2}}\,ds + \int_0^b \frac{1}{\sqrt{4-s^2}}\,ds\right]\\ &= -\lim_{b\to 2^-}\left[\sqrt{4-s^2}\right]_0^b + \lim_{b\to 2^-}\left[sen^{-1}\left(\frac{s}{2}\right)\right]_0^b\\ &= -\lim_{b\to 2^-}\left[\sqrt{4-b^2}-2\right] + \lim_{b\to 2^-}\left[sen^{-1}\left(\frac{b}{2}\right) - \underbrace{sen^{-1}0}_{0}\right]\\ &= -\underbrace{\left(\lim_{b\to 2^-}\sqrt{4-b^2}\right)}_{0} + 2 + \underbrace{\left(\lim_{b\to 2^-} sen^{-1}\left(\frac{b}{2}\right)\right)}_{\pi/2} = 2+\frac{\pi}{2}.\end{aligned}$$

(f)

$$\begin{aligned}\int_{-1}^{8}\frac{dx}{x^{1/3}} &= \int_{-1}^{0}\frac{dx}{x^{1/3}} + \int_0^8 \frac{dx}{x^{1/3}}\\ &= \lim_{b\to 0^-}\int_{-1}^{b}\frac{dx}{x^{1/3}} + \lim_{a\to 0^+}\int_a^8\frac{dx}{x^{1/3}} = \frac{3}{2}\lim_{b\to 0^-}\left[x^{2/3}\right]_{-1}^{b} + \frac{3}{2}\lim_{a\to 0^+}\left[x^{2/3}\right]_a^8\\ &= \frac{3}{2}\lim_{b\to 0^-}\left[b^{2/3}-(-1)^{2/3}\right] + \frac{3}{2}\lim_{a\to 0^+}\left[8^{2/3}-a^{2/3}\right]\\ &= \frac{3}{2}\underbrace{\left(\lim_{b\to 0^-} b^{2/3}\right)}_{0} - \frac{3}{2} + \frac{12}{2} - \frac{3}{2}\underbrace{\left(\lim_{a\to 0^+} a^{2/3}\right)}_{0} = \frac{9}{2}.\end{aligned}$$

(g)

$$\begin{aligned}
\int_{-1}^{1} \frac{dx}{x^5} &= \int_{-1}^{0} \frac{dx}{x^5} + \int_{0}^{1} \frac{dx}{x^5} \\
&= \lim_{b\to 0^-} \int_{-1}^{b} \frac{dx}{x^5} + \lim_{a\to 0^+} \int_{a}^{1} \frac{dx}{x^5} = -\frac{1}{4} \lim_{b\to 0^-} \left[\frac{1}{x^4}\right]_{-1}^{b} - \frac{1}{4} \lim_{a\to 0^+} \left[\frac{1}{x^4}\right]_{a}^{1} \\
&= -\frac{1}{4} \lim_{b\to 0^-} \left[\frac{1}{b^4} - 1\right] - \frac{1}{4} \lim_{a\to 0^+} \left[1 - \frac{1}{a^4}\right] \\
&= -\frac{1}{4} \underbrace{\left(\lim_{b\to 0^-} \frac{1}{b^4}\right)}_{\infty} + \frac{1}{4} \underbrace{\left(\lim_{a\to 0^+} \frac{1}{a^4}\right)}_{\infty} = -\infty + \infty \quad \therefore \text{ diverge.}
\end{aligned}$$

(h)

$$\begin{aligned}
\int_{e^{-1}}^{e} \frac{dx}{x(\ln x)^3} &= \int_{-1}^{1} \frac{du}{u^3} = \int_{-1}^{0} \frac{du}{u^3} + \int_{0}^{1} \frac{du}{u^3} \\
&= \lim_{b\to 0^-} \int_{-1}^{b} \frac{du}{u^3} + \lim_{a\to 0^+} \int_{a}^{1} \frac{du}{u^3} = \lim_{b\to 0^-} \left[-\frac{1}{2u^2}\right]_{-1}^{b} + \lim_{a\to 0^+} \left[-\frac{1}{2u^2}\right]_{a}^{1} \\
&= \lim_{b\to 0^-} \left[-\frac{1}{2b^2} + \frac{1}{2}\right] + \lim_{a\to 0^+} \left[-\frac{1}{2} + \frac{1}{2a^2}\right] \quad \therefore \text{ diverge.}
\end{aligned}$$

(i)

$$\begin{aligned}
\int_{0}^{2} \frac{dx}{\sqrt{|x-1|}} &= \int_{0}^{1} \frac{dx}{\sqrt{1-x}} + \int_{1}^{2} \frac{dx}{\sqrt{x-1}} \\
&= \lim_{b\to 1^-} \int_{0}^{b} \frac{dx}{\sqrt{1-x}} + \lim_{a\to 1^+} \int_{a}^{2} \frac{dx}{\sqrt{x-1}} \\
&= -2 \lim_{b\to 1^-} \left[\sqrt{1-x}\right]_{0}^{b} + 2 \lim_{a\to 1^+} \left[\sqrt{x-1}\right]_{a}^{2} \\
&= -2 \lim_{b\to 1^-} \left[\sqrt{1-b} - 1\right] + 2 \lim_{a\to 1^+} \left[1 - \sqrt{a-1}\right] \\
&= -2 \underbrace{\left(\lim_{b\to 1^-} \sqrt{1-b}\right)}_{0} + 2 + 2 - 2 \underbrace{\left(\lim_{a\to 1^+} \sqrt{a-1}\right)}_{0} = 4.
\end{aligned}$$

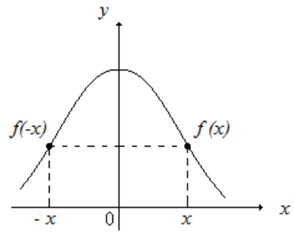
3. (a) Efectuando el cambio de variable $u = \ln x$ se obtiene

$$\int_{1}^{e} \frac{dx}{x(\ln x)^p} = \int_{0}^{1} \frac{du}{u^p} = \begin{cases} \dfrac{1}{1-p}, & \text{si } p < 1, \\ \text{diverge}, & \text{si } p \geq 1. \end{cases}$$

(b) Efectuando el cambio de variable $u = \ln x$ se obtiene

$$\int_{e}^{\infty} \frac{dx}{x(\ln x)^p} = \int_{1}^{\infty} \frac{du}{u^p} = \begin{cases} \text{diverge}, & \text{si } p \leq 1, \\ \dfrac{1}{p-1}, & \text{si } p > 1. \end{cases}$$

4. (a) Si f es par entonces $f(-x) = f(x)$.



Partimos de

$$\int_{-\infty}^{\infty} f(x)\ dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\ dx.$$

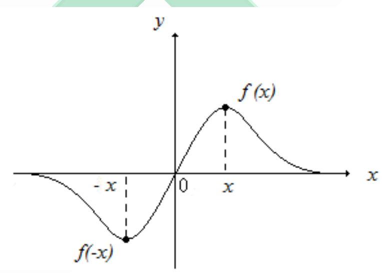
Ahora reescribimos $\int_{-\infty}^{0} f(x)\ dx$. Para ello, utilizamos el hecho de que $f(-x) = f(x)$ e introducimos el cambio de variable $u = -x$, de modo que

$$\int_{-\infty}^{0} f(x)\,dx = \int_{-\infty}^{0} f(-x)\ dx = \int_{\infty}^{0} f(u)\ (-du) = \int_{0}^{\infty} f(u)\ du.$$

De esta manera,

$$\int_{-\infty}^{\infty} f(x)\ dx = \underbrace{\int_{0}^{\infty} f(u)\ du + \int_{0}^{\infty} f(x)\ dx}_{\text{son iguales}} = 2\int_{0}^{\infty} f(x)\ dx.$$

(b) Si f es impar entonces $f(-x) = -f(x)$.



Partimos de

$$\int_{-\infty}^{\infty} f(x)\ dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\ dx.$$

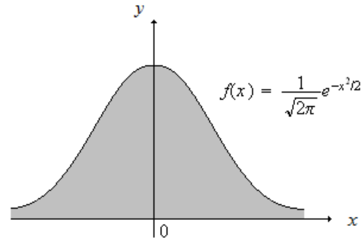
Ahora reescribimos $\int_{-\infty}^{0} f(x)\ dx$. Para ello, utilizamos el hecho de que $-f(-x) = f(x)$ e introducimos el cambio de variable $u = -x$, de modo que

$$\int_{-\infty}^{0} f(x)\,dx = -\int_{-\infty}^{0} f(-x)\ dx = -\int_{\infty}^{0} f(u)\ (-du) = -\int_{0}^{\infty} f(u)\ du.$$

De esta manera,

$$\int_{-\infty}^{\infty} f(x)\ dx = \underbrace{-\int_{0}^{\infty} f(u)\ du + \int_{0}^{\infty} f(x)\ dx}_{\text{son iguales}} = 0.$$

5. Sea $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ en $(-\infty, \infty)$, con $\int_{-\infty}^{\infty} f(x)\ dx = 1$. Como $f(x) = f(-x)$, por lo tanto f es una función par.



(a) Calculemos la media $\mu = \int\limits_{-\infty}^{\infty} x f(x)\ dx$. Sabemos que

$$\mu = \int\limits_{-\infty}^{\infty} x f(x)\ dx = \int\limits_{-\infty}^{0} x f(x)\ dx + \int\limits_{0}^{\infty} x f(x)\ dx.$$

Ahora reescribimos $\int_{-\infty}^{0} x f(x)\ dx$. Para ello, utilizamos el hecho de que $f(-x) = f(x)$ e introducimos el cambio de variable $u = -x$, de modo que

$$\int\limits_{-\infty}^{0} x f(x)\ dx = \int\limits_{-\infty}^{0} x f(-x)\ dx = \int\limits_{\infty}^{0} (-u) f(u)\ (-du) = \int\limits_{\infty}^{0} u f(u)\ du = -\int\limits_{0}^{\infty} u f(u)\ du.$$

De esta manera,

$$\mu = -\int\limits_{0}^{\infty} u f(u)\ du + \int\limits_{0}^{\infty} x f(x)\ dx = 0.$$

(b) Calculemos la varianza $\sigma^2 = \int\limits_{-\infty}^{\infty} x^2 f(x)\ dx - \left(\int\limits_{-\infty}^{\infty} x f(x)\ dx \right)^2$. Tomando en cuenta el inciso anterior ($\mu = 0$), se tiene

$$\sigma^2 = \int\limits_{-\infty}^{\infty} x^2 f(x)\ dx - \left(\int\limits_{-\infty}^{\infty} x f(x)\ dx \right)^2 = \int\limits_{-\infty}^{\infty} x^2 f(x)\ dx - \mu^2 = \int\limits_{-\infty}^{\infty} x^2 f(x)\ dx.$$

Partiendo la integral resultante, se tiene

$$\sigma^2 = \int\limits_{-\infty}^{0} x^2 f(x)\ dx + \int\limits_{0}^{\infty} x^2 f(x)\ dx.$$

Ahora reescribimos $\int_{-\infty}^{0} x^2 f(x)\ dx$. Para ello, utilizamos el hecho de que $f(-x) = f(x)$ e introducimos el cambio de variable $u = -x$, de modo que

$$\int\limits_{-\infty}^{0} x^2 f(x)\ dx = \int\limits_{-\infty}^{0} x^2 f(-x)\ dx = \int\limits_{\infty}^{0} (-u)^2 f(u)\ (-du) = \int\limits_{0}^{\infty} u^2 f(u)\ du.$$

De esta manera,

$$\sigma^2 = \int\limits_{-\infty}^{\infty} x^2 f(x)\ dx = \int\limits_{0}^{\infty} u^2 f(u)\ du + \int\limits_{0}^{\infty} x^2 f(x)\ dx = 2 \int\limits_{0}^{\infty} x^2 f(x)\ dx.$$

Por último, calculamos la integral resultante:

$$\begin{aligned}\sigma^2 &= 2\int_0^\infty x^2 f(x)\ dx = 2\lim_{b\to\infty}\int_0^b x^2 f(x)\ dx = \frac{2}{\sqrt{2\pi}}\lim_{b\to\infty}\int_0^b x^2 e^{-x^2/2}\ dx \\ &= \frac{2}{\sqrt{2\pi}}\lim_{b\to\infty}\int_0^b \underbrace{x}_{u}\underbrace{\left(xe^{-x^2/2}\right)\ dx}_{dv} = \frac{2}{\sqrt{2\pi}}\lim_{b\to\infty}\left[\left[-xe^{-x^2/2}\right]_0^b + \int_0^b e^{-x^2/2}\ dx\right] \\ &= -\frac{2}{\sqrt{2\pi}}\underbrace{\left(\lim_{b\to\infty} be^{-b^2/2}\right)}_{0\ (\text{*L'Hopital})} + 2\int_0^\infty \frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx = \underbrace{\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx}_{1\ (\text{ver enunciado})} = 1.\end{aligned}$$

$$\text{* L'Hopital:}\quad \lim_{b\to\infty} be^{-b^2/2} = \lim_{b\to\infty}\frac{b}{e^{b^2/2}} \overset{L}{=} \lim_{b\to\infty}\frac{1}{be^{b^2/2}} = 0.$$

6. Sea $f(x) = \frac{1}{a}e^{-x/a}$ en $[0,\infty)$, con $a > 0$.

(a)

$$\begin{aligned}\int_0^\infty f(x)\ dx &= \frac{1}{a}\lim_{b\to\infty}\int_0^b e^{-x/a}\ dx = \frac{1}{a}\lim_{b\to\infty}\left[-ae^{-x/a}\right]_0^b = -\lim_{b\to\infty}\left[e^{-x/a}\right]_0^b \\ &= -\lim_{b\to\infty}\left[e^{-b/a}-1\right] = -\underbrace{\left(\lim_{b\to\infty}\frac{1}{e^{b/a}}\right)}_{0\ (a>0)} + 1 = 1.\end{aligned}$$

(b)

$$\begin{aligned}\int_0^\infty xf(x)\ dx &= \frac{1}{a}\lim_{b\to\infty}\int_0^b xe^{-x/a}\ dx = \frac{1}{a}\lim_{b\to\infty}\left[-axe^{-x/a} - a^2e^{-x/a}\right]_0^b \\ &= \lim_{b\to\infty}\left[-xe^{-x/a} - ae^{-x/a}\right]_0^b = \lim_{b\to\infty}\left[\left(-be^{-b/a} - ae^{-b/a}\right) - (0-a)\right] \\ &= -\underbrace{\left(\lim_{b\to\infty}\frac{b}{e^{b/a}}\right)}_{0\ (\text{*L'Hopital})} - a\underbrace{\left(\lim_{b\to\infty}\frac{1}{e^{b/a}}\right)}_{0\ (a>0)} + a = a.\end{aligned}$$

$$\text{* L'Hopital:}\quad \lim_{b\to\infty}\frac{b}{e^{b/a}} \overset{L}{=} \lim_{b\to\infty}\frac{1}{\frac{1}{a}e^{b/a}} = 0.$$

(c)

$$\int_0^\infty (x-a)f(x)\ dx = \underbrace{\int_0^\infty xf(x)\ dx}_{a} - a\underbrace{\int_0^\infty f(x)\ dx}_{1} = a - a(1) = 0.$$

(d)

$$\begin{aligned}
\int_0^\infty x^2 f(x)\ dx &= \frac{1}{a}\lim_{b\to\infty}\int_0^b x^2 e^{-x/a}\ dx = \frac{1}{a}\lim_{b\to\infty}\left[-ax^2e^{-x/a}-2a^2xe^{-x/a}-2a^3e^{-x/a}\right]_0^b \\
&= \lim_{b\to\infty}\left[-x^2e^{-x/a}-2axe^{-x/a}-2a^2e^{-x/a}\right]_0^b \\
&= \lim_{b\to\infty}\left[\left(-b^2e^{-b/a}-2abe^{-b/a}-2a^2e^{-b/a}\right)+2a^2\right] \\
&= -\underbrace{\left(\lim_{b\to\infty}\frac{b^2}{e^{b/a}}\right)}_{0\text{ (* L'Hopital)}} - 2a\underbrace{\left(\lim_{b\to\infty}\frac{b}{e^{b/a}}\right)}_{0\text{ (** L'Hopital)}} - 2a^2\underbrace{\left(\lim_{b\to\infty}\frac{1}{e^{b/a}}\right)}_{0\ (a>0)} + 2a^2 = 2a^2.
\end{aligned}$$

$$\text{* L'Hopital:}\quad \lim_{b\to\infty}\frac{b^2}{e^{b/a}} \overset{L}{=} \lim_{b\to\infty}\frac{2b}{\frac{1}{a}e^{b/a}} \overset{L}{=} \lim_{b\to\infty}\frac{2}{\frac{1}{a^2}e^{b/a}} = 0.$$

$$\text{** L'Hopital:}\quad \lim_{b\to\infty}\frac{b}{e^{b/a}} \overset{L}{=} \lim_{b\to\infty}\frac{1}{\frac{1}{a}e^{b/a}} = 0.$$

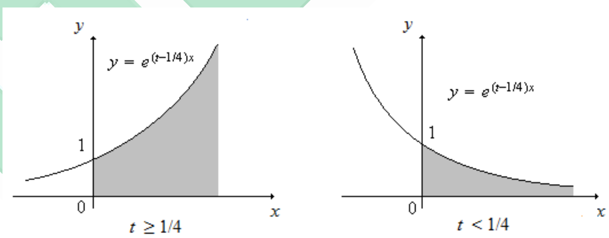
(e)

$$\begin{aligned}
\int_0^\infty (x-a)^2 f(x)\ dx &= \int_0^\infty \left(x^2-2ax+a^2\right) f(x)\, dx \\
&= \underbrace{\int_0^\infty x^2 f(x)\ dx}_{2a^2} - 2a\underbrace{\int_0^\infty xf(x)\ dx}_{a} + a^2\underbrace{\int_0^\infty f(x)\ dx}_{1} = a^2.
\end{aligned}$$

7. Observa que

$$m(t) = \int_0^\infty e^{tx}\left(\frac{e^{-x/4}}{4}\right)\ dx = \frac{1}{4}\int_0^\infty e^{(t-1/4)x}\ dx,$$

por lo que se debe analizar por separado los casos $t \geq 1/4$ y $t < 1/4$.



i) Si $t \geq 1/4$, entonces la integral diverge (ver figura de la izquierda).

ii) Si $t < \frac{1}{4}$ (figura de la derecha), entonces

$$\begin{aligned}
m(t) &= \frac{1}{4}\lim_{b\to\infty}\int_0^b e^{(t-1/4)x}\ dx = \frac{1}{4(t-1/4)}\lim_{b\to\infty}\int_0^{(t-1/4)b} e^u\ du \\
&= \frac{1}{4t-1}\lim_{b\to\infty}\left[e^u\right]_0^{(t-1/4)b} = \frac{1}{4t-1}\lim_{b\to\infty}\left[e^{(t-1/4)b}-1\right] \\
&= \frac{1}{4t-1}\underbrace{\left(\lim_{b\to\infty}e^{(t-1/4)b}\right)}_{0\ (t<1/4)} - \frac{1}{4t-1} = \frac{1}{1-4t}.
\end{aligned}$$

Concluimos entonces que

$$m(t) = \int_0^\infty e^{tx}\left(\frac{e^{-x/4}}{4}\right)\,dx = \begin{cases} \dfrac{1}{1-4t}, & t < \frac{1}{4}, \\ \text{diverge}, & t \geq \frac{1}{4}. \end{cases}$$

8. Sea $\Gamma(\alpha) = \int_0^\infty y^{\alpha-1}e^{-y}\,dy$, $\alpha \in \mathbb{R}^+$.

(a)

$$\begin{aligned}\Gamma(1) &= \int_0^\infty \underbrace{y^{1-1}}_{1}\, e^{-y}\,dy = \int_0^\infty e^{-y}\,dy = \lim_{a\to\infty}\int_0^a e^{-y}\,dy = \lim_{a\to\infty}\left[-e^{-y}\right]_0^a \\ &= \lim_{a\to\infty}\left[-e^{-a}+1\right] = -\underbrace{\left(\lim_{a\to\infty} e^{-a}\right)}_{0} + 1 = 1.\end{aligned}$$

(b)

$$\begin{aligned}\Gamma(\alpha) &= \int_0^\infty y^{\alpha-1}e^{-y}\,dy = \lim_{b\to\infty}\int_0^b y^{\alpha-1}e^{-y}\,dy \\ &= \lim_{b\to\infty}\left\{\left[-y^{\alpha-1}e^{-y}\right]_0^b + (\alpha-1)\int_0^b y^{\alpha-2}e^{-y}dy\right\} \\ &= -\underbrace{\left(\lim_{b\to\infty} b^{\alpha-1}e^{-b}\right)}_{0\ (\text{* L´Hopital})} + (\alpha-1)\underbrace{\int_0^\infty y^{(\alpha-1)-1}e^{-y}dy}_{\Gamma(\alpha-1)} = (\alpha-1)\Gamma(\alpha-1).\end{aligned}$$

* L'Hopital: $\lim_{b\to\infty}\dfrac{b^{\alpha-1}}{e^b} \overset{L}{=} \lim_{b\to\infty}\dfrac{(\alpha-1)\,b^{\alpha-2}}{e^b} \overset{L}{=} \ldots \overset{L}{=} \lim_{b\to\infty}\dfrac{(\alpha-1)(\alpha-2)\cdot\ldots\cdot(1)}{e^b} = 0.$

(c) Sabemos que $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$ y que $\Gamma(1) = 1$. De esta manera,

$$\begin{aligned}\Gamma(1) &= 1 &&= 0! \\ \Gamma(2) &= 1\cdot\Gamma(1) &&= 1\cdot 1 &&= 1! \\ \Gamma(3) &= 2\cdot\Gamma(2) &&= 2\cdot 1 &&= 2! \\ \Gamma(4) &= 3\cdot\Gamma(3) &&= 3\,(2!) &&= 3!, \text{ etc...}\end{aligned}$$

Por lo tanto,

$$\Gamma(n) = (n-1)!, \qquad n \in \mathbb{Z}^+.$$

9. Sea $f(y) = \dfrac{y^{\alpha-1}e^{-y/\beta}}{\beta^\alpha\Gamma(\alpha)}$ en $[0,\infty)$.

(a) Efectuando el cambio de variable $u = y/\beta$ se tiene

$$\int_0^\infty f(y)\,dy = \frac{1}{\beta^\alpha\Gamma(\alpha)}\int_0^\infty y^{\alpha-1}e^{-y/\beta}\,dy = \frac{1}{\Gamma(\alpha)}\underbrace{\int_0^\infty u^{\alpha-1}e^{-u}\,du}_{\Gamma(\alpha)} = \frac{1}{\Gamma(\alpha)}\Gamma(\alpha) = 1.$$

(b) Efectuando el cambio de variable $u = y/\beta$ se tiene

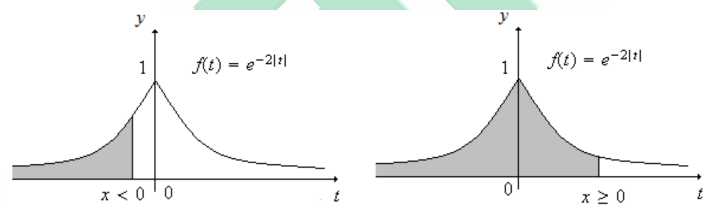
$$\begin{aligned}\int_0^{\infty} yf(y)\, dy &= \frac{1}{\beta^{\alpha}\Gamma(\alpha)}\int_0^{\infty} y\left(y^{\alpha-1}e^{-y/\beta}\right)\, dy = \frac{\beta}{\Gamma(\alpha)}\int_0^{\infty} u^{\alpha}e^{-u}\, du \\ &= \frac{\beta}{\Gamma(\alpha)}\int_0^{\infty} u^{(\alpha+1)-1}e^{-u}\, du = \frac{\beta}{\Gamma(\alpha)}\underbrace{\Gamma(\alpha+1)}_{\alpha\Gamma(\alpha)} = \alpha\beta.\end{aligned}$$

(c) Efectuando el cambio de variable $u = y/\beta$ se tiene

$$\begin{aligned}\int_0^{\infty} y^2 f(y)\, dy &= \frac{1}{\beta^{\alpha}\Gamma(\alpha)}\int_0^{\infty} y^2\left(y^{\alpha-1}e^{-y/\beta}\right)\, dy = \frac{\beta^2}{\Gamma(\alpha)}\int_0^{\infty} u^{\alpha+1}e^{-u}du \\ &= \frac{\beta^2}{\Gamma(\alpha)}\int_0^{\infty} u^{(\alpha+2)-1}e^{-u}du = \frac{\beta^2}{\Gamma(\alpha)}\underbrace{\Gamma(\alpha+2)}_{(\alpha+1)\Gamma(\alpha+1)} \\ &= \beta^2(\alpha+1)\frac{\overbrace{\Gamma(\alpha+1)}^{\alpha\Gamma(\alpha)}}{\Gamma(\alpha)} = \alpha(\alpha+1)\beta^2.\end{aligned}$$

10. La función de distribución acumulada para la función de densidad $f(x) = e^{-2|x|}$ es

$$F_X(x) = \int_{-\infty}^{x} f(t)\, dt = \int_{-\infty}^{x} e^{-2|t|}\, dt, \quad x \in \mathbb{R}.$$



Si $x < 0$, entonces

$$\begin{aligned}F_X(x) &= \int_{-\infty}^{x} e^{-2(-t)}\, dt = \int_{-\infty}^{x} e^{2t}\, dt = \lim_{a\to-\infty}\int_a^x e^{2t}\, dt = \frac{1}{2}\lim_{a\to-\infty}\left[e^{2t}\right]_a^x \\ &= \frac{1}{2}\lim_{a\to-\infty}\left(e^{2x} - e^{2a}\right) = \frac{1}{2}e^{2x}.\end{aligned}$$

Si $x \geq 0$, entonces

$$\begin{aligned}F_X(x) &= \int_{-\infty}^{0} e^{-2(-t)}\, dt + \int_0^x e^{-2(t)}\, dt = \lim_{a\to-\infty}\int_a^0 e^{2t}\, dt + \int_0^x e^{-2t}\, dt \\ &= \frac{1}{2}\lim_{a\to-\infty}\left[e^{2t}\right]_a^0 - \frac{1}{2}\left[e^{-2t}\right]_0^x = \frac{1}{2} - \frac{1}{2}\left(e^{-2x} - 1\right) = 1 - \frac{1}{2}e^{-2x}.\end{aligned}$$

Por lo tanto,

$$F_X(x) = \begin{cases} \frac{1}{2}e^{2x}, & \text{si } x < 0 \\ 1 - \frac{1}{2}e^{-2x} & \text{si } x \geq 0. \end{cases}$$

CÁLCULO III
TAREA 9 - SOLUCIONES
INTEGRACIÓN MÚLTIPLE
(Temas 3.1-3.7)

1. (a)

$$\begin{aligned}\int_0^3\int_{-1}^0 \left(x^2y-2xy\right)\,dydx &= \int_0^3\int_{-1}^0 \left(x^2-2x\right)y\,dydx=\int_0^3 \left(x^2-2x\right)\left[\frac{y^2}{2}\right]_{-1}^0 dx\\ &= \int_0^3 \left(x^2-2x\right)\left[\frac{0^2}{2}-\frac{(-1)^2}{2}\right]dx=\left(-\frac{1}{2}\right)\int_0^3 \left(x^2-2x\right)dx\\ &= \left(-\frac{1}{2}\right)\left[\frac{x^3}{3}-x^2\right]_0^3=\left(-\frac{1}{2}\right)\left(\frac{3^3}{3}-3^2\right)=0.\end{aligned}$$

(b)

$$\begin{aligned}\int_0^3\int_0^1 2x\sqrt{x^2+y}\,dxdy &= \int_0^3 \frac{2}{3}\left[\left(x^2+y\right)^{3/2}\right]_0^1 dy=\frac{2}{3}\int_0^3\left[(y+1)^{3/2}-y^{3/2}\right]dy\\ &= \frac{2}{3}\left\{\frac{2}{5}\left[(y+1)^{5/2}\right]_0^3-\frac{2}{5}\left[y^{5/2}\right]_0^3\right\}\\ &= \frac{4}{15}\left[\underbrace{4^{5/2}}_{32}-1\right]-\frac{4}{15}\left[3^{5/2}\right]=\frac{4}{15}\left[31-3^{5/2}\right].\end{aligned}$$

(c)

$$\begin{aligned}\int_0^{\ln 3}\int_0^{\ln 2} e^{2x+y}\,dydx &= \int_0^{\ln 3} e^{2x}\int_0^{\ln 2} e^y\,dydx=\int_0^{\ln 3} e^{2x}\left[e^y\right]_0^{\ln 2}\,dx\\ &= \int_0^{\ln 3} e^{2x}\left[\underbrace{e^{\ln 2}}_{2}-\underbrace{e^0}_{1}\right]dx=\int_0^{\ln 3} e^{2x}\,dx=\frac{1}{2}\left[e^{2x}\right]_0^{\ln 3}\\ &= \frac{1}{2}\left[e^{2\ln 3}-e^0\right]=\frac{1}{2}\left[e^{\ln 3^2}-1\right]=\frac{1}{2}\left[9-1\right]=4.\end{aligned}$$

(d)

$$\int_0^1\int_0^1 xe^{xy}\,dydx=\int_0^1 \left[e^{xy}\right]_0^1\,dx=\int_0^1\left[e^x-1\right]dx=\left[e^x-x\right]_0^1=\left(e^1-1\right)-e^0=e-2.$$

2. (a)

$$\begin{aligned}\iint_R \left(\left(3^x+1\right)^2-1\right)dA &= \int_0^1\int_0^{\ln 3}\left[3^{2x}+2\left(3^x\right)\right]dydx=\int_0^1\left[3^{2x}+2\left(3^x\right)\right]\left[y\right]_0^{\ln 3}dx\\ &= \ln 3\int_0^1\left(3^{2x}+2\left(3^x\right)\right)dx=\ln 3\left[\frac{1}{2\ln 3}3^{2x}+\frac{2}{\ln 3}\left(3^x\right)\right]_0^1\\ &= \left[\frac{1}{2}3^{2x}+2\left(3^x\right)\right]_0^1=\left[\left(\frac{1}{2}9+2\left(3\right)\right)-\left(\frac{1}{2}+2\right)\right]=8.\end{aligned}$$

(b)

$$\iint_R \frac{e^{4x}}{x}\,dA = \int_1^2 \int_{-2x}^{2x} \frac{e^{4x}}{x}\,dydx = \int_1^2 \frac{e^{4x}}{x}\,[y]_{-2x}^{2x}\,dx = \int_1^2 \frac{e^{4x}}{x}\,(4x)\,dx = \int_1^2 4e^{4x}\,dx$$

$$= \left[e^{4x}\right]_1^2 = e^8 - e^4.$$

(c)

$$\iint_R e^{x^2}\,dA = \int_0^1 \int_{1-x}^{1+x} e^{x^2}\,dydx = \int_0^1 e^{x^2}\,[y]_{1-x}^{1+x}\,dx = \int_0^1 2xe^{x^2}dx = \left[e^{x^2}\right]_0^1 = e - 1.$$

(d)

$$\iint_R \frac{1}{1+y^2}\,dA = \int_0^1 \int_y^{3y} \frac{1}{1+y^2}\,dxdy = \int_0^1 \left(\frac{1}{1+y^2}\right)[x]_y^{3y}\,dy = \int_0^1 \frac{2y}{1+y^2}\,dy$$

$$= \left[\ln\left(1+y^2\right)\right]_0^1 = \ln 2 - \ln 1 = \ln 2.$$

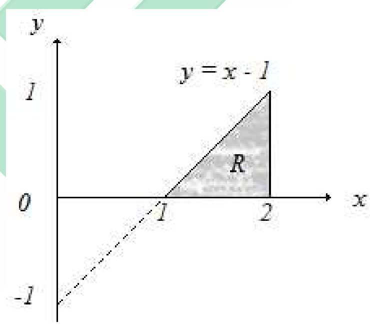
(e)

$$\iint_R \frac{1}{2x+x^2}\,dA = \int_1^3 \int_0^{1+x} \frac{1}{2x+x^2}\,dydx = \int_1^3 \frac{1}{2x+x^2}\,[y]_0^{1+x}\,dx$$

$$= \int_1^3 \frac{1+x}{2x+x^2}\,dx = \frac{1}{2}\int_1^3 \frac{2+2x}{2x+x^2}\,dx = \frac{1}{2}\left[\ln\left(2x+x^2\right)\right]_1^3$$

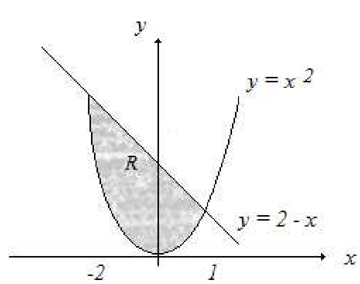
$$= \frac{1}{2}\left[\ln 15 - \ln 3\right] = \frac{1}{2}\ln\left(\frac{15}{3}\right) = \frac{1}{2}\ln 5.$$

3. (a)



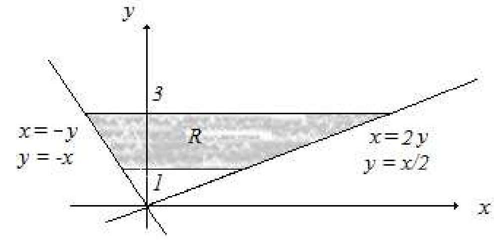
$$\int_1^2 \int_0^{x-1} y\,dydx = \int_1^2 \left[\frac{y^2}{2}\right]_0^{x-1} dx = \frac{1}{2}\int_1^2 (x-1)^2\,dx = \frac{1}{6}\left[(x-1)^3\right]_1^2 = \frac{1}{6}.$$

(b)



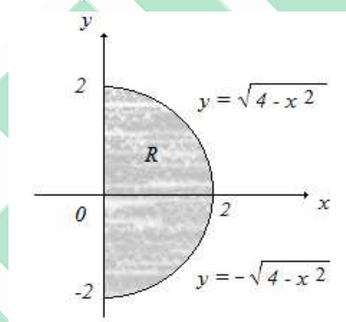
$$\int_{-2}^{1}\int_{x^2}^{2-x} xy\,dydx = \int_{-2}^{1} x\left[\frac{y^2}{2}\right]_{x^2}^{2-x} dx = \frac{1}{2}\int_{-2}^{1} x\left[(2-x)^2 - x^4\right]dx$$
$$= \frac{1}{2}\int_{-2}^{1}\left[4x - 4x^2 + x^3 - x^5\right]dx = \left[x^2 - \frac{2}{3}x^3 + \frac{x^4}{8} - \frac{x^6}{12}\right]_{-2}^{1} = -\frac{45}{8}.$$

(c)



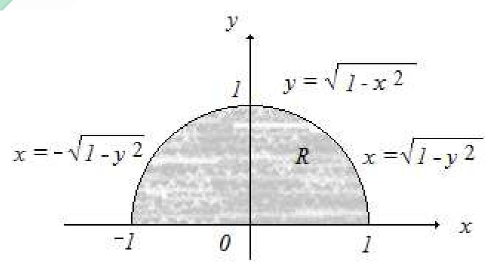
$$\int_{1}^{3}\int_{-y}^{2y} xe^{y^3}\,dxdy = \int_{1}^{3} e^{y^3}\left[\frac{x^2}{2}\right]_{-y}^{2y} dy = \frac{1}{2}\int_{1}^{3} e^{y^3}\left[4y^2 - y^2\right]dy$$
$$= \frac{1}{2}\int_{1}^{3} 3y^2e^{y^3}\,dy = \frac{1}{2}\left[e^{y^3}\right]_{1}^{3} = \frac{1}{2}\left[e^{27} - e\right].$$

(d)



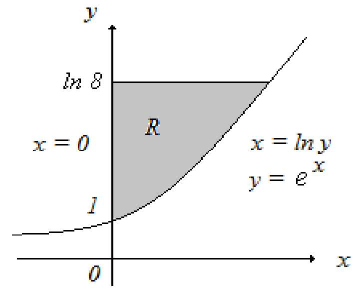
$$\int_{0}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} x\,dydx = \int_{0}^{2} x\left[y\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx = \int_{0}^{2} 2x\sqrt{4-x^2}\,dx = -\frac{2}{3}\left[\left(4-x^2\right)^{3/2}\right]_{0}^{2} = \frac{16}{3}.$$

(e)



$$\int_{0}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} y\,dxdy = \int_{0}^{1} y\left[x\right]_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dy = \int_{0}^{1} 2y\sqrt{1-y^2}\,dy = -\frac{2}{3}\left[\left(1-y^2\right)^{3/2}\right]_{0}^{1} = \frac{2}{3}.$$

(f)

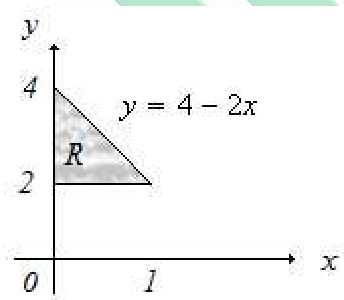


$$\int_1^{\ln 8}\int_0^{\ln y} e^{x+y}\,dxdy = \int_1^{\ln 8} e^y \int_0^{\ln y} e^x\,dxdy = \int_1^{\ln 8} e^y\left[e^x\right]_0^{\ln y}\,dy = \int_1^{\ln 8} e^y\left[\underbrace{e^{\ln y}}_{y}-1\right]dy$$

$$= \int_1^{\ln 8}\underbrace{e^y\left[y-1\right]}_{\text{int. por partes}}dy = \left[(y-1)\,e^y - e^y\right]_1^{\ln 8} = \left[(y-2)\,e^y\right]_1^{\ln 8}$$

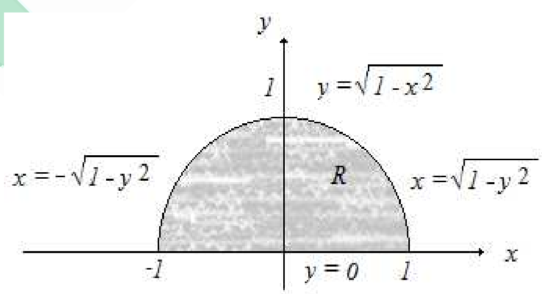
$$= (\ln 8 - 2)\underbrace{e^{\ln 8}}_{8} - (1-2)\,e^1 = 8\ln 8 - 16 + e.$$

4. (a)



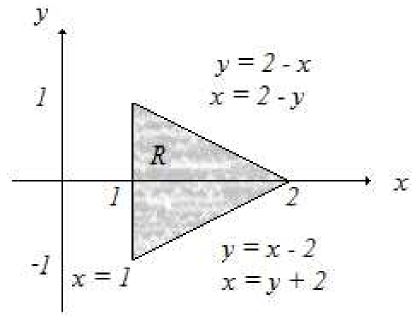
$$\int_0^1\int_2^{4-2x} f(x,y)\,dydx = \int_2^4\int_0^{2-y/2} f(x,y)\,dxdy.$$

(b)



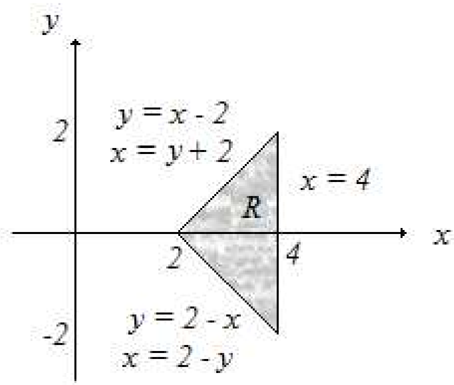
$$\int_0^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\,dxdy = \int_{-1}^1\int_0^{\sqrt{1-x^2}} f(x,y)\,dydx.$$

(c)



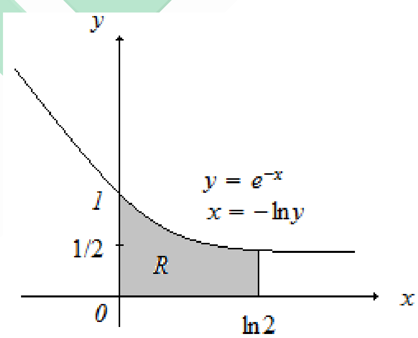
$$\int_{1}^{2}\int_{x-2}^{2-x} f(x,y)\ dydx = \int_{-1}^{0}\int_{1}^{y+2} f(x,y)\ dxdy + \int_{0}^{1}\int_{1}^{2-y} f(x,y)\ dxdy.$$

(d)



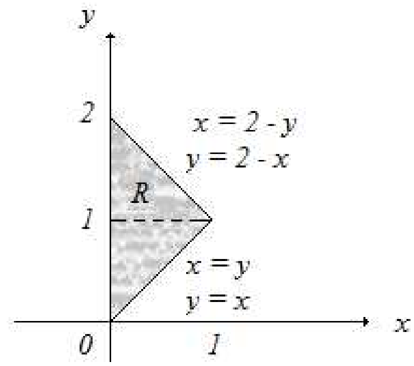
$$\int_{2}^{4}\int_{2-x}^{x-2} f(x,y)\ dydx = \int_{-2}^{0}\int_{2-y}^{4} f(x,y)\ dxdy + \int_{0}^{2}\int_{y+2}^{4} f(x,y)\ dxdy.$$

(e)



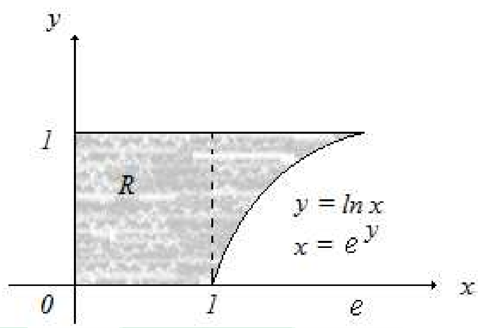
$$\int_{0}^{\ln 2}\int_{0}^{e^{-x}} f(x,y)\ dydx = \int_{0}^{1/2}\int_{0}^{\ln 2} f(x,y)\ dxdy + \int_{1/2}^{1}\int_{0}^{-\ln y} f(x,y)\ dxdy.$$

(f)



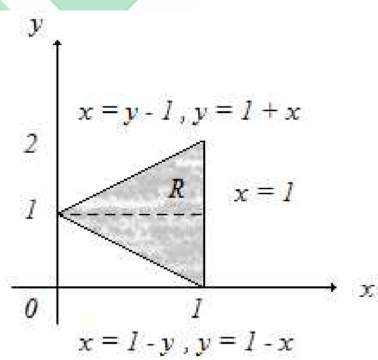
$$\int_0^1 \int_0^y f(x,y)\ dxdy + \int_1^2 \int_0^{2-y} f(x,y)\ dxdy = \int_0^1 \int_x^{2-x} f(x,y)\ dydx.$$

(g)



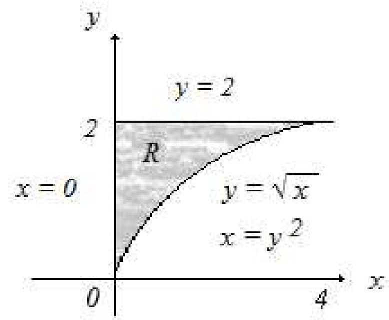
$$\int_0^1 \int_0^1 f(x,y)\ dydx + \int_1^e \int_{\ln x}^1 f(x,y)\ dydx = \int_0^1 \int_0^{e^y} f(x,y)\ dxdy.$$

(h)



$$\int_0^1 \int_{1-y}^1 f(x,y)\ dxdy + \int_1^2 \int_{y-1}^1 f(x,y)\ dxdy = \int_0^1 \int_{1-x}^{1+x} f(x,y)\ dydx.$$

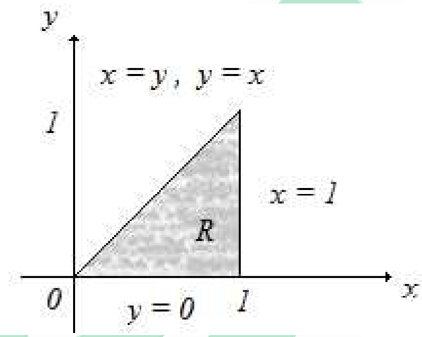
5. (a)



$$\int_0^4 \int_{\sqrt{x}}^2 \frac{3}{y^3+1}\, dydx = \int_0^2 \int_0^{y^2} \frac{3}{y^3+1}\, dxdy = \int_0^2 \frac{3}{y^3+1}\, [x]_0^{y^2}\, dy = \int_0^2 \frac{3y^2}{y^3+1}\, dy$$

$$= \left[\ln\left(y^3+1\right)\right]_0^2 = \ln 9 - \ln 1 = \ln 9.$$

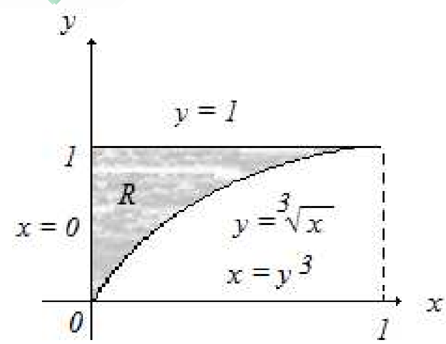
(b)



$$\int_0^1 \int_y^1 \frac{2x}{x^3+1}\, dxdy = \int_0^1 \int_0^x \frac{2x}{x^3+1}\, dydx = \int_0^1 \frac{2x}{x^3+1}\, [y]_0^x\, dx = 2\int_0^1 \frac{x^2}{x^3+1} dx$$

$$= \frac{2}{3}\left[\ln\left(x^3+1\right)\right]_0^1 = \frac{2}{3}\left[\ln 2 - \ln 1\right] = \frac{2}{3}\ln 2.$$

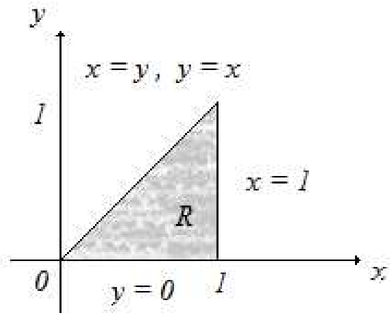
(c)



$$\int_0^1 \int_{\sqrt[3]{x}}^1 4y^4 e^{xy}\, dydx = \int_0^1 \int_0^{y^3} 4y^4 e^{xy}\, dxdy = \int_0^1 4y^3 \int_0^{y^3} y e^{xy}\, dxdy = \int_0^1 4y^3 \left[e^{xy}\right]_{x=0}^{x=y^3}\, dy$$

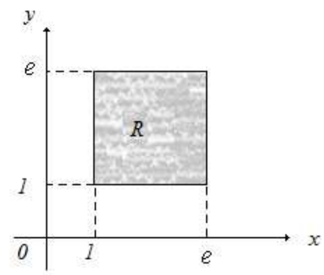
$$= \int_0^1 4y^3 \left[e^{y^4} - 1\right] dy = \left[e^{y^4} - y^4\right]_0^1 = (e-1) - (1-0) = e - 2.$$

(d)



$$\int_0^1\int_y^1 e^{x^2}\,dxdy=\int_0^1\int_0^x e^{x^2}\,dydx=\int_0^1 e^{x^2}\,[y]_0^x\,dx=\int_0^1 xe^{x^2}dx=\frac{1}{2}\left[e^{x^2}\right]_0^1=\frac{1}{2}\left[e-1\right].$$

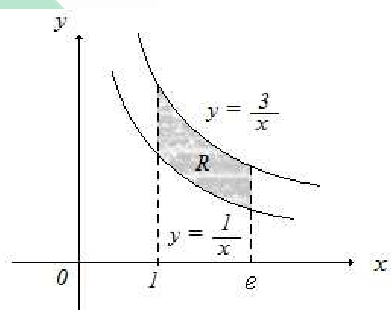
6. (a)



$$A_R=\int_1^e\int_1^e dydx=(e-1)^2\,.$$

$$\begin{aligned}\overline{f}&=\frac{1}{A_R}\int_1^e\int_1^e\frac{1}{xy}\,dydx=\frac{1}{A_R}\int_1^e\frac{1}{x}\,[\ln|y|]_1^e\,dx=\frac{1}{A_R}\int_1^e\frac{1}{x}\left[\underbrace{\ln e}_{1}-\underbrace{\ln 1}_{0}\right]dx\\&=\frac{1}{A_R}\,[\ln|x|]_1^e=\frac{1}{A_R}\,[\ln e-\ln 1]=\frac{1}{A_R}=\frac{1}{(e-1)^2}.\end{aligned}$$

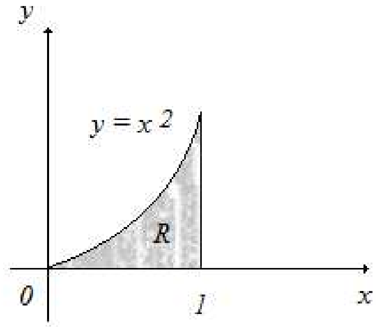
(b)



$$A_R=\int_1^e\int_{1/x}^{3/x}dydx=\int_1^e\frac{2}{x}\,dx=2\,[\ln|x|]_1^e=2\,(\ln e-\ln 1)=2.$$

$$\begin{aligned}\overline{f}&=\frac{1}{A_R}\int_1^e\int_{1/x}^{3/x}e^{xy}\,dydx=\frac{1}{A_R}\int_1^e\left[\frac{e^{xy}}{x}\right]_{y=1/x}^{y=3/x}dx=\frac{1}{A_R}\int_1^e\frac{1}{x}\left[e^3-e\right]dx\\&=\frac{e^3-e}{A_R}\,[\ln|x|]_1^e=\frac{e^3-e}{A_R}\,(\ln e-\ln 1)=\frac{e^3-e}{2}.\end{aligned}$$

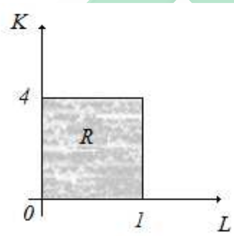
(c)



$$A_R = \int_0^1 \int_0^{x^2} dydx = \int_0^1 x^2\, dx = \frac{1}{3}\left[x^3\right]_0^1 = \frac{1}{3}.$$

$$\begin{aligned}\overline{f} &= \frac{1}{A_R}\int_0^1 \int_0^{x^2} 3x^3 e^{xy}\, dydx = \frac{1}{A_R}\int_0^1 3x^2 \int_0^{x^2} xe^{xy}\, dydx = \frac{1}{A_R}\int_0^1 3x^2\left[e^{xy}\right]_{y=0}^{y=x^2}\, dx \\ &= \frac{1}{A_R}\int_0^1 3x^2\left[e^{x^3}-1\right] dx = \frac{1}{A_R}\left[e^{x^3}-x^3\right]_0^1 = \frac{e-2}{A_R} = 3\left(e-2\right).\end{aligned}$$

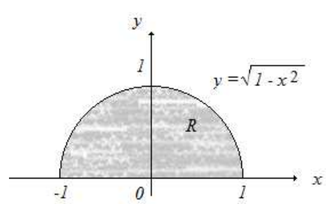
7.



$$A_R = \int_0^1 \int_0^4 dKdL = 4.$$

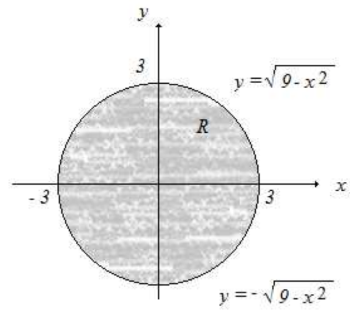
$$\begin{aligned}\overline{P} &= \frac{1}{A_R}\int_0^1 \int_0^4 L^{1/2}K^{1/2}\, dKdL = \frac{1}{A_R}\int_0^1 L^{1/2}\underbrace{\left[\frac{2}{3}K^{3/2}\right]_0^4}_{16/3} dL = \frac{16}{3A_R}\int_0^1 L^{1/2}\, dL \\ &= \frac{16}{3A_R}\underbrace{\left[\frac{2}{3}L^{3/2}\right]_0^1}_{2/3} = \frac{32}{9A_R} = \frac{8}{9}.\end{aligned}$$

8. (a)



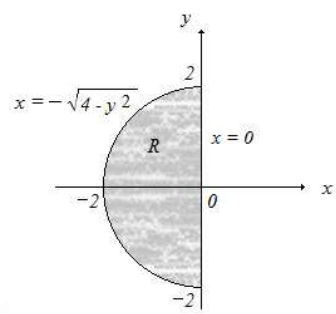
$$\int_{-1}^1 \int_0^{\sqrt{1-x^2}} dydx = \int_0^\pi \int_0^1 rdrd\theta = \int_0^\pi \left[\frac{r^2}{2}\right]_0^1 d\theta = \frac{1}{2}\int_0^\pi d\theta = \frac{1}{2}\left[\theta\right]_0^\pi = \frac{\pi}{2}.$$

(b)



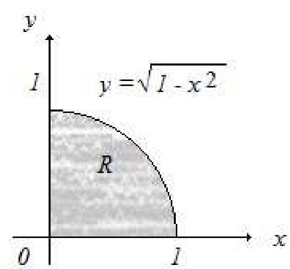
$$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dydx = \int_0^{2\pi}\int_0^3 rdrd\theta = \int_0^{2\pi}\left[\frac{r^2}{2}\right]_0^3 d\theta = \frac{9}{2}\int_0^{2\pi} d\theta = \frac{9}{2}(2\pi) = 9\pi.$$

(c)



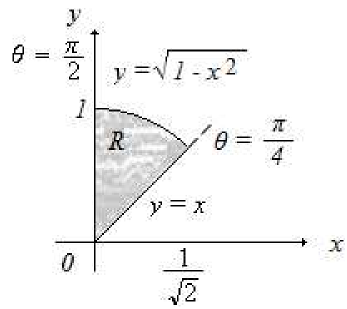
$$\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{0}\left(x^2+y^2\right) dxdy = \int_{\pi/2}^{3\pi/2}\int_0^2 r^2\, rdrd\theta = \int_{\pi/2}^{3\pi/2}\left[\frac{r^4}{4}\right]_0^2 d\theta = 4\int_{\pi/2}^{3\pi/2} d\theta = 4\pi.$$

(d)



$$\begin{aligned}\int_0^1\int_0^{\sqrt{1-x^2}} e^{-\left(x^2+y^2\right)} dydx &= \int_0^{\pi/2}\int_0^1 e^{-r^2}\, rdrd\theta = \int_0^{\pi/2}\left[-\frac{1}{2}e^{-r^2}\right]_0^1 d\theta \\ &= \left(-\frac{1}{2}e^{-1}+\frac{1}{2}\right)\int_0^{\pi/2} d\theta = \left(-\frac{1}{2}e^{-1}+\frac{1}{2}\right)\frac{\pi}{2} = \frac{\pi\left(1-e^{-1}\right)}{4}.\end{aligned}$$

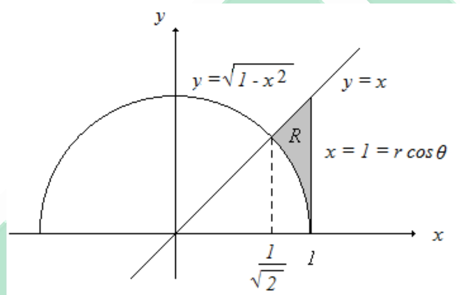
(e)



$$\int_0^{1/\sqrt{2}} \int_x^{\sqrt{1-x^2}} e^{-\sqrt{x^2+y^2}}\, dydx = \int_{\pi/4}^{\pi/2} \int_0^1 e^{-r}\, rdrd\theta = \int_{\pi/4}^{\pi/2} \left[-(r+1)\,e^{-r}\right]_0^1 d\theta$$

$$= \left(-2e^{-1}+1\right) \int_{\pi/4}^{\pi/2} d\theta = \left(-2e^{-1}+1\right)\frac{\pi}{4} = \frac{\pi\left(-2e^{-1}+1\right)}{4}.$$

(f)



$$\int_{1/\sqrt{2}}^{1} \int_{\sqrt{1-x^2}}^{x} 8xy\, dydx = \int_0^{\pi/4} \int_1^{1/\cos\theta} 8\,(r\cos\theta)\,(r\,sen\,\theta)\; rdrd\theta$$

$$= \int_0^{\pi/4} 8\cos\theta\, sen\,\theta \left[\frac{r^4}{4}\right]_1^{1/\cos\theta} d\theta$$

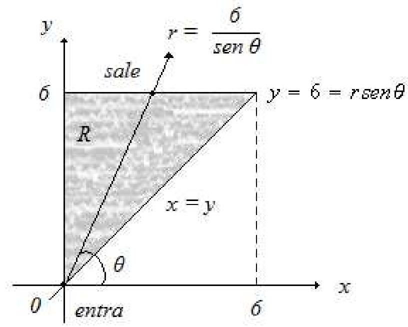
$$= \int_0^{\pi/4} 2\cos\theta\, sen\,\theta \left[\left(\frac{1}{\cos\theta}\right)^4 - 1\right] d\theta$$

$$= \int_0^{\pi/4} \left[\frac{2\,sen\,\theta}{\cos^3\theta} - 2\,sen\,\theta\cos\theta\right] d\theta = \left[\frac{1}{\cos^2\theta} + \cos^2\theta\right]_0^{\pi/4}$$

$$= \frac{1}{\cos^2(\pi/4)} + \cos^2(\pi/4) - \frac{1}{\cos^2(0)} - \cos^2(0)$$

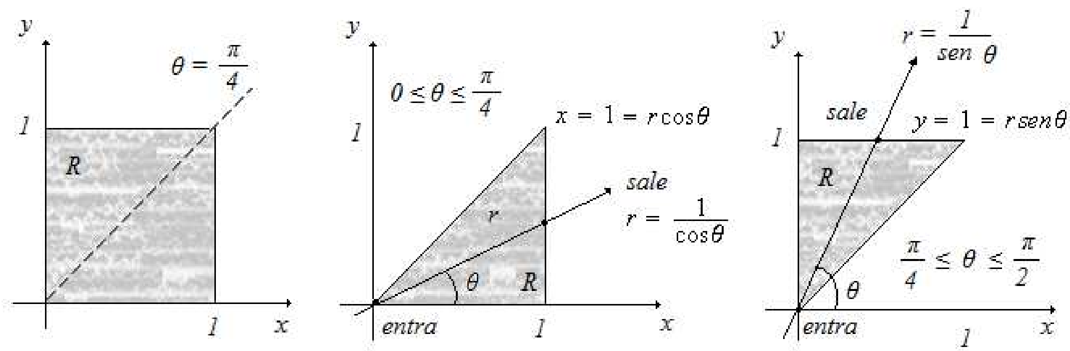
$$= \frac{1}{\left(1/\sqrt{2}\right)^2} + \left(1/\sqrt{2}\right)^2 - 1 - 1 = \frac{1}{2}.$$

(g)



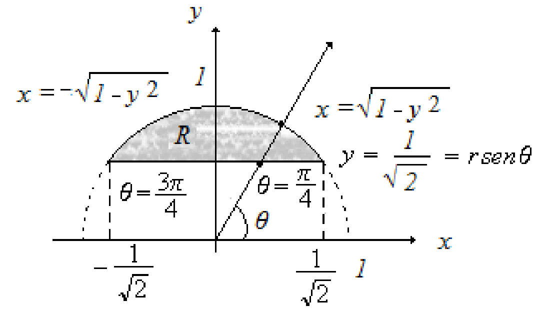
$$
\begin{aligned}
\int_0^6\int_0^y x\,dxdy &= \int_{\pi/4}^{\pi/2}\int_0^{6/sen\,\theta} r\cos\theta\, rdrd\theta = \int_{\pi/4}^{\pi/2}\cos\theta\left[\frac{r^3}{3}\right]_0^{6/sen\,\theta} d\theta\\
&= \frac{1}{3}\int_{\pi/4}^{\pi/2}\cos\theta\left(\frac{6^3}{sen^3\theta}\right)d\theta = \frac{216}{3}\int_{\pi/4}^{\pi/2}(sen\theta)^{-3}\cos\theta\, d\theta\\
&= \frac{216}{3}\int_{1/\sqrt{2}}^{1} u^{-3}\,du = \frac{216}{3}\left[-\frac{u^{-2}}{2}\right]_{1/\sqrt{2}}^{1}\\
&= -\frac{108}{3}\left[1-\frac{1}{(1/\sqrt{2})^2}\right] = -36\,(1-2) = 36.
\end{aligned}
$$

(h)



$$
\begin{aligned}
\int_0^1\int_0^1 dxdy &= \int_0^{\pi/4}\int_0^{\frac{1}{\cos\theta}=\sec\theta} r\,drd\theta + \int_{\pi/4}^{\pi/2}\int_0^{\frac{1}{sen\theta}=\csc\theta} r\,drd\theta\\
&= \int_0^{\pi/4}\left[\frac{r^2}{2}\right]_0^{\sec\theta} d\theta + \int_{\pi/4}^{\pi/2}\left[\frac{r^2}{2}\right]_0^{\csc\theta} d\theta = \frac{1}{2}\left[\int_0^{\pi/4}\sec^2\theta\,d\theta + \int_{\pi/4}^{\pi/2}\csc^2\theta\,d\theta\right]\\
&= \frac{1}{2}\left[\tan\theta\right]_0^{\pi/4} - \frac{1}{2}\left[\cot\theta\right]_{\pi/4}^{\pi/2} = \frac{1}{2}\left[\underbrace{\tan\frac{\pi}{4}}_{1} - \underbrace{\tan 0}_{0}\right] - \frac{1}{2}\left[\underbrace{\cot\frac{\pi}{2}}_{0} - \underbrace{\cot\frac{\pi}{4}}_{1}\right] = 1.
\end{aligned}
$$

(i)

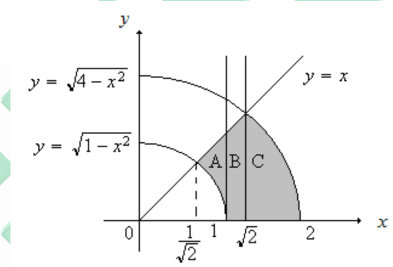


$$\int_{1/\sqrt{2}}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dxdy = \int_{\pi/4}^{3\pi/4}\int_{\frac{1}{\sqrt{2}}\csc\theta}^{1} rdrd\theta = \int_{\pi/4}^{3\pi/4}\left[\frac{r^2}{2}\right]_{\frac{1}{\sqrt{2}}\csc\theta}^{1} d\theta$$

$$= \int_{\pi/4}^{3\pi/4}\left[\frac{1}{2}-\frac{1}{4}\csc^2\theta\right] d\theta = \left[\frac{\theta}{2}+\frac{1}{4}\cot\theta\right]_{\pi/4}^{3\pi/4}$$

$$= \frac{3\pi}{8}+\frac{1}{4}\underbrace{\cot\left(\frac{3\pi}{4}\right)}_{-1}-\frac{\pi}{8}-\frac{1}{4}\underbrace{\cot\frac{\pi}{4}}_{1} = \frac{1}{2}\left[\frac{\pi}{2}-1\right].$$

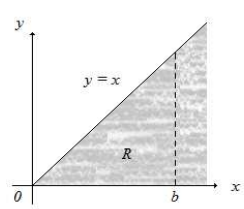
(j)



$$\underbrace{\int_{1/\sqrt{2}}^{1}\int_{\sqrt{1-x^2}}^{x} dydx}_{\text{A}} + \underbrace{\int_{1}^{\sqrt{2}}\int_{0}^{x} dydx}_{\text{B}} + \underbrace{\int_{\sqrt{2}}^{2}\int_{0}^{\sqrt{4-x^2}} dydx}_{\text{C}} = \int_{0}^{\pi/4}\int_{1}^{2} rdrd\theta = \int_{0}^{\pi/4}\left[\frac{r^2}{2}\right]_{1}^{2} d\theta$$

$$= \frac{3}{2}\int_{0}^{\pi/4} d\theta = \frac{3}{2}\left(\frac{\pi}{4}\right) = \frac{3\pi}{8}.$$

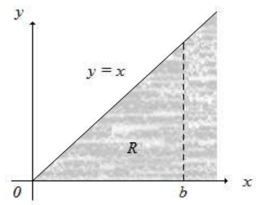
9. (a)



$$\int_{0}^{\infty}\int_{0}^{x} e^{-x^2}\,dydx = \lim_{b\to\infty}\int_{0}^{b}\int_{0}^{x} e^{-x^2}\,dydx = \lim_{b\to\infty}\int_{0}^{b} e^{-x^2}\,[y]_0^x\,dx = \lim_{b\to\infty}\int_{0}^{b} xe^{-x^2}\,dx$$

$$= -\frac{1}{2}\lim_{b\to\infty}\left[e^{-x^2}\right]_0^b = -\frac{1}{2}\lim_{b\to\infty}\left(e^{-b^2}-1\right) = \frac{1}{2}.$$

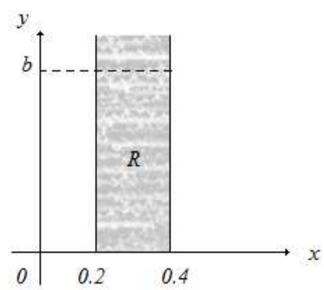
(b)



$$
\begin{aligned}
\int_0^\infty \int_0^x e^{-x}\, dydx &= \lim_{b\to\infty} \int_0^b \int_0^x e^{-x}\, dydx = \lim_{b\to\infty} \int_0^b e^{-x}\,[y]_0^x\, dx = \lim_{b\to\infty} \int_0^b xe^{-x}\, dx \\
&= \lim_{b\to\infty} \left[-xe^{-x} - e^{-x}\right]_0^b = \lim_{b\to\infty} \left[-be^{-b} - e^{-b} + 1\right] \\
&= -\underbrace{\left(\lim_{b\to\infty} \frac{b}{e^b}\right)}_{0\ (\text{*L´Hopital})} - \underbrace{\left(\lim_{b\to\infty} \frac{1}{e^b}\right)}_{0} + 1 = 1.
\end{aligned}
$$

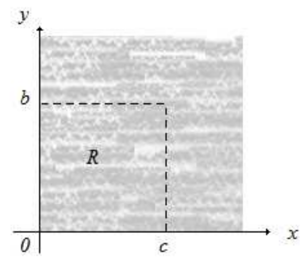
$$
\text{*L'Hopital:} \quad \lim_{b\to\infty} \frac{b}{e^b} \overset{L}{=} \lim_{b\to\infty} \frac{1}{e^b} = 0.
$$

(c)



$$
\begin{aligned}
\underbrace{\int_0^\infty \int_{0.2}^{0.4}}_{\text{Región rectangular}} 5xe^{-xy}\, dxdy &= \int_{0.2}^{0.4} \int_0^\infty 5xe^{-xy}\, dydx = 5 \int_{0.2}^{0.4} \lim_{b\to\infty} \int_0^b xe^{-xy}\, dydx \\
&= -5 \int_{0.2}^{0.4} \lim_{b\to\infty} \left[e^{-xy}\right]_{y=0}^{y=b} dx = -5 \int_{0.2}^{0.4} \lim_{b\to\infty} \left[e^{-bx} - 1\right] dx \\
&= -5 \lim_{b\to\infty} \left[-\frac{1}{b} e^{-bx} - x\right]_{0.2}^{0.4} \\
&= 5 \left\{ \underbrace{\lim_{b\to\infty} \left(\frac{1}{be^{0.4b}}\right)}_{0} + 0.4 - \underbrace{\lim_{b\to\infty} \left(\frac{1}{be^{0.2b}}\right)}_{0} - 0.2 \right\} = 1.
\end{aligned}
$$

(d)

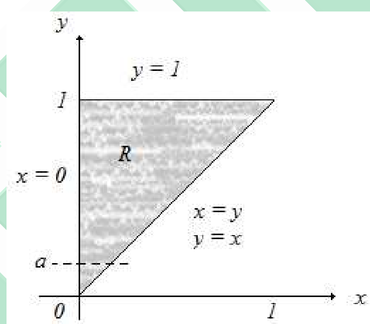


$$\int_0^\infty \int_0^\infty xe^{-(x+2y)}\,dxdy = \underbrace{\lim_{b\to\infty}\int_0^b \lim_{c\to\infty}\int_0^c}_{\text{Región Rectangular}} \underbrace{xe^{-x}e^{-2y}}_{\text{Integrando factorizable}}\,dxdy$$

$$= \left[\lim_{b\to\infty}\int_0^b \underbrace{xe^{-x}dx}_{\text{por partes}}\right]\left[\lim_{c\to\infty}\int_0^c e^{-2y}dy\right]$$

$$= \underbrace{\lim_{b\to\infty}\left[-xe^{-x}-e^{-x}\right]_0^b}_{1\ \ (\text{ver 9b})} \underbrace{\lim_{c\to\infty}\left[-\frac{1}{2}e^{-2y}\right]_0^c}_{1/2} = \frac{1}{2}.$$

(e)



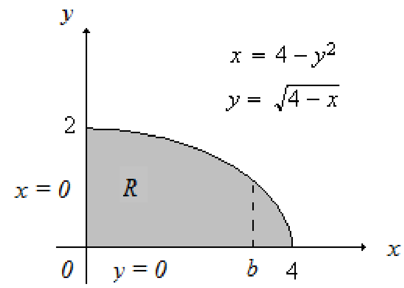
$$\int_0^1\int_x^1 \frac{sen\,y}{y}\,dydx = \int_0^1\int_0^y \frac{sen\,y}{y}\,dxdy \underset{\text{int. impropia}}{=} \lim_{a\to 0^+}\int_a^1\int_0^y \frac{sen\,y}{y}\,dxdy$$

$$= \lim_{a\to 0^+}\int_a^1 \left(\frac{sen\,y}{y}\right)[x]_0^y\,dy = \lim_{a\to 0^+}\int_a^1 \frac{sen\,y}{y}(y-0)\,dy$$

$$= \lim_{a\to 0^+}\int_a^1 sen\,y\,dy \underset{\text{int. propia}}{=} \int_0^1 sen\,y\,dy = [-\cos y]_0^1$$

$$= -\cos 1 + \cos 0 = 1 - \cos 1.$$

(f)



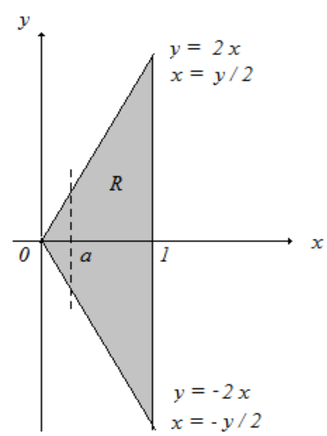
$$\int_0^2 \int_0^{4-y^2} \frac{2ye^x}{4-x}\,dxdy = \int_0^4 \int_0^{\sqrt{4-x}} \frac{2ye^x}{4-x}\,dydx = \lim_{b\to 4^-} \int_0^b \int_0^{\sqrt{4-x}} \frac{2ye^x}{4-x}\,dydx$$

$$= \lim_{b\to 4^-} \int_0^b \frac{e^x}{4-x} \int_0^{\sqrt{4-x}} 2y\,dydx = \lim_{b\to 4^-} \int_0^b \frac{e^x}{4-x}\left[y^2\right]_0^{\sqrt{4-x}}\,dx$$

$$= \lim_{b\to 4^-} \int_0^b \frac{e^x}{4-x}(4-x)\,dx = \lim_{b\to 4^-} \int_0^b e^x\,dx = \int_0^4 e^x\,dx$$

$$= \left[e^x\right]_0^4 = e^4 - 1.$$

(g)



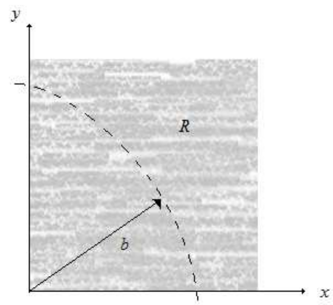
$$\int_{-2}^0 \int_{-y/2}^1 \frac{e^{4x}}{x}\,dxdy + \int_0^2 \int_{y/2}^1 \frac{e^{4x}}{x}\,dxdy = \int_0^1 \int_{-2x}^{2x} \frac{e^{4x}}{x}\,dydx = \lim_{a\to 0^+} \int_a^1 \int_{-2x}^{2x} \frac{e^{4x}}{x}\,dydx$$

$$= \lim_{a\to 0^+} \int_a^1 \frac{e^{4x}}{x}\left[y\right]_{-2x}^{2x}\,dx = \lim_{a\to 0^+} \int_a^1 \frac{e^{4x}}{x}(4x)\,dx$$

$$= \lim_{a\to 0^+} \int_a^1 4e^{4x}\,dx = \int_0^1 4e^{4x}\,dx$$

$$= \left[e^{4x}\right]_0^1 = e^4 - 1.$$

10. (a)

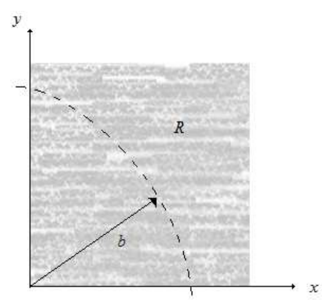


$$\int_0^{\infty}\int_0^{\infty} e^{-\left(x^2+y^2\right)}\,dydx = \int_0^{\pi/2}\int_0^{\infty} e^{-r^2}\,rdrd\theta = \int_0^{\pi/2}\left(\lim_{b\to\infty}\int_0^{b} e^{-r^2}\,rdr\right)d\theta$$

$$= \int_0^{\pi/2}\lim_{b\to\infty}\left[-\frac{1}{2}e^{-r^2}\right]_{r=0}^{r=b} d\theta = \int_0^{\pi/2}\left[-\frac{1}{2}\underbrace{\left(\lim_{b\to\infty}\ e^{-b^2}\right)}_{0}+\frac{1}{2}\right]d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} d\theta = \frac{1}{2}\left(\frac{\pi}{2}\right) = \frac{\pi}{4}.$$

(b)



$$\int_0^{\infty}\int_0^{\infty}\frac{1}{\left(1+x^2+y^2\right)^2}dydx = \int_0^{\pi/2}\int_0^{\infty}\frac{1}{\left(1+r^2\right)^2}rdrd\theta = \int_0^{\pi/2}\left(\lim_{b\to\infty}\int_0^{b}\frac{r}{\left(1+r^2\right)^2}dr\right)d\theta$$

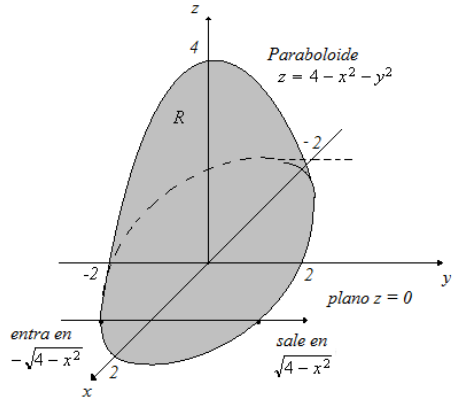
$$= \int_0^{\pi/2}\lim_{b\to\infty}\left[-\frac{1}{2\left(1+r^2\right)}\right]_{r=0}^{r=b} d\theta$$

$$= \int_0^{\pi/2}\left[-\frac{1}{2}\underbrace{\left(\lim_{b\to\infty}\ \frac{1}{1+b^2}\right)}_{0}+\frac{1}{2}\right]d\theta = \frac{1}{2}\int_0^{\pi/2} d\theta = \frac{1}{2}\left(\frac{\pi}{2}\right) = \frac{\pi}{4}.$$

11.

$$\int_1^{e}\int_1^{e}\int_1^{e}\frac{1}{xyz}\,dxdydz = \int_1^{e}\int_1^{e}\frac{1}{yz}\underbrace{\left[\ln|x|\right]_1^e}_{1}dydz = \int_1^{e}\frac{1}{z}\underbrace{\left[\ln|y|\right]_1^e}_{1}dz = \left[\ln|z|\right]_1^e = 1.$$

12.



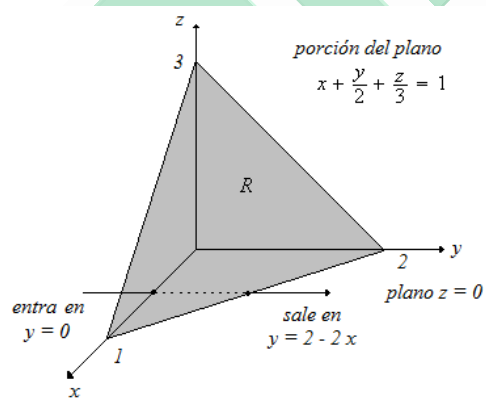
Las superficies $z = 0$ y $z = 4 - x^2 - y^2$ se intersecan en $4 - x^2 - y^2 = 0$, es decir,

$$x^2 + y^2 = 4, \quad \text{con } -2 \leq x \leq 2.$$

Por lo tanto, el volumen V de la región R está dado por

$$V = \int_{x=-2}^{x=2} \int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}} \int_{z=0}^{z=4-x^2-y^2} 1 \, dzdydx.$$

13.



Las superficies $z = 0$ y $x + \dfrac{y}{2} + \dfrac{z}{3} = 1$ se intersecan en $x + \dfrac{y}{2} = 1$, $x, y \geq 0$, es decir,

$$y = 2 - 2x, \quad \text{con } 0 \leq x \leq 1.$$

Por lo tanto, el volumen V de la región R está dado por

$$V = \int_{x=0}^{x=1} \int_{y=0}^{y=2-2x} \int_{z=0}^{z=3-3x-\frac{3}{2}y} 1 \, dzdydx.$$

CÁLCULO III
TAREA 10
SUCESIONES - SOLUCIONES
(Temas 4.1-4.3)

1. (a) $a_n = \dfrac{1}{n!}$

   Se trata de la sucesión $1, \dfrac{1}{2!}, \dfrac{1}{3!}, \dfrac{1}{4!}, \dfrac{1}{5!}, ... = 1, \dfrac{1}{2}, \dfrac{1}{6}, \dfrac{1}{24}, \dfrac{1}{120}, ...$

   (b) $a_n = 2 + (-1)^n$.

   Se trata de la sucesión $1, 3, 1, 3, 1, ...$

   (c) $a_n = \dfrac{2^n}{2^{n+1}} = \dfrac{1}{2}$.

   Se trata de la sucesión $\dfrac{1}{2}, \dfrac{1}{2}, \dfrac{1}{2}, \dfrac{1}{2}, \dfrac{1}{2}, ...$

   (d) $a_n = \dfrac{1-n}{n^2}$.

   Se trata de la sucesión $0, -\dfrac{1}{4}, -\dfrac{2}{9}, -\dfrac{3}{16}, -\dfrac{4}{25}, ...$

   (e) $a_{n+1} = (n+1)\, a_n, \quad a_1 = 1$.

   Como

$$\begin{array}{lllllll} a_1 & = & 1 & = & 1! & & \\ a_2 & = & (1+1)\, a_1 & = & 2a_1 & = & 2 & = & 2! \\ a_3 & = & (2+1)\, a_2 & = & 3a_2 & = & 3 \cdot 2 & = & 3! \\ a_4 & = & (3+1)\, a_3 & = & 4a_3 & = & 4 \cdot 3 \cdot 2 & = & 4! \\ a_5 & = & (4+1)\, a_4 & = & 5a_4 & = & 5 \cdot 4 \cdot 3 \cdot 2 & = & 5!, \ \text{etc...} \end{array}$$

   por lo tanto se trata de la sucesión $1!, 2!, 3!, 4!, 5!, ... = 1, 2, 6, 24, 120, ...$

   (f) $a_{n+2} = \dfrac{a_{n+1}}{a_n}, \quad a_1 = 2, \quad a_2 = -1$.

   Como

$$\begin{array}{lllllll} a_1 & = & 2 & & & & \\ a_2 & = & -1 & & & & \\ a_3 & = & \dfrac{a_2}{a_1} & = & \dfrac{(-1)}{2} & = & -\dfrac{1}{2} \\ a_4 & = & \dfrac{a_3}{a_2} & = & \dfrac{(-1/2)}{(-1)} & = & \dfrac{1}{2} \\ a_5 & = & \dfrac{a_4}{a_3} & = & \dfrac{(1/2)}{(-1/2)} & = & -1, \ \text{etc...} \end{array}$$

   por lo tanto se trata de la sucesión $2, -1, -\dfrac{1}{2}, \dfrac{1}{2}, -1, ...$

2. (a) $1, -1, 1, -1, 1, ...$

   Una posible respuesta es $a_n = (-1)^{n+1}$, $n \geq 1$.

   (b) $-1, 1, -1, 1, -1...$

   Una posible respuesta es $a_n = (-1)^n$, $n \geq 1$.

(c) $1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}, \ldots$

Una posible respuesta es $a_n = (-1)^{n+1} \frac{1}{n^2}$, $n \geq 1$.

(d) $-3, -2, -1, 0, 1, 2, \ldots$

Una posible respuesta es $a_n = n - 4$, $n \geq 1$.

(e) $1, -3, 5, -7, 9, -11, \ldots$

Una posible respuesta es $a_n = (-1)^{n+1} (2n - 1)$, $n \geq 1$.

(f) $2, 6, 10, 14, 18, \ldots$

Una posible respuesta es $a_n = 2(2n - 1) = 4n - 2$, $n \geq 1$.

(g) $1, 0, 1, 0, 1, 0, \ldots$

Una posible respuesta es $a_n = \frac{1}{2}\left(1 + (-1)^{n+1}\right) = \frac{1 + (-1)^{n+1}}{2}$, $n \geq 1$.

(h) $0, 1, 1, 2, 2, 3, 3, 4, 4, 5, 5, \ldots$

Una posible respuesta es $a_n = \frac{1}{2}\left(n - \frac{1 + (-1)^{n+1}}{2}\right)$, $n \geq 0$.

Otra respuesta puede ser $a_n = n - \left\lfloor \frac{n}{2} \right\rfloor$, $n \geq 0$, en donde $\lfloor x \rfloor$ es la "función piso" de x (mayor entero menor o igual que x).

3. (a) $a_n = \ln(3e^n) - n$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} [\ln(3e^n) - n] = \lim_{n\to\infty} [\ln 3 + \ln e^n - n] = \lim_{n\to\infty} [\ln 3 + n - n] = \ln 3.$$

(b) $a_n = \frac{sen\, n}{n}$.

Usaremos el teorema del sándwich:

$$\begin{aligned} -1 &\leq sen\, n \leq 1 \\ -\frac{1}{n} &\leq \frac{sen\, n}{n} \leq \frac{1}{n} \\ -\lim_{n\to\infty}\left(\frac{1}{n}\right) &\leq \lim_{n\to\infty}\left(\frac{sen\, n}{n}\right) \leq \lim_{n\to\infty}\left(\frac{1}{n}\right) \\ 0 &\leq \lim_{n\to\infty}\left(\frac{sen\, n}{n}\right) \leq 0. \end{aligned}$$

Por lo tanto,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{sen\, n}{n} = 0.$$

(c) $a_n = ((-1)^n + 1)\left(\frac{n+1}{n}\right)$.

Aquí no se cumplen las hipótesis del teorema del sándwich. La sucesión no tiene un límite, ya que

$$\begin{aligned} &k \text{ par} &&\Longrightarrow && a_k = 2\left(\frac{k+1}{k}\right) = 2 + \frac{2}{k} &&\Longrightarrow && \lim_{k\to\infty} a_k = 2 \\ &k \text{ impar} &&\Longrightarrow && a_k = 0 &&\Longrightarrow && \lim_{k\to\infty} a_k = 0. \end{aligned}$$

Por lo tanto,

$$\lim_{n\to\infty} a_n \quad \text{no existe}$$
$$\therefore \quad \{a_n\} \quad \text{diverge.}$$

(d) $a_n = (-1)^n \left(\dfrac{n}{n^2+1}\right)$.

Podemos utilizar el teorema del valor absoluto:

Como
$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{1}{n+\frac{1}{n}} = 0,$$
por lo tanto
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left((-1)^n \frac{n}{n^2+1}\right) = 0.$$
Nota que en este ejemplo también puede utilizarse el teorema del sándwich.

(e) $a_n = (-1)^n \left(\dfrac{3n^2}{n^2+1}\right)$.

Aquí no podemos utilizar el teorema del valor absoluto, ya que $\lim_{n\to\infty} |a_n| \neq 0$.
Tampoco se cumplen las hipótesis del teorema del sándwich.
La sucesión no tiene un límite, ya que

$$k \text{ par} \implies a_k = \frac{3k^2}{k^2+1} = \frac{3}{1+\frac{1}{k^2}} \implies \lim_{k\to\infty} a_k = 3$$
$$k \text{ impar} \implies a_k = -\frac{3k^2}{k^2+1} = -\frac{3}{1+\frac{1}{k^2}} \implies \lim_{k\to\infty} a_k = -3.$$

Por lo tanto,

$$\lim_{n\to\infty} a_n \quad \text{no existe}$$
$$\therefore \quad \{a_n\} \quad \text{diverge.}$$

(f) $a_n = \dfrac{3^n}{n^3}$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3^n}{n^3} \stackrel{L}{=} \lim_{n\to\infty} \frac{3^n \ln 3}{3n^2} \stackrel{L}{=} \lim_{n\to\infty} \frac{3^n (\ln 3)^2}{6n} \stackrel{L}{=} \lim_{n\to\infty} \frac{3^n (\ln 3)^3}{6} = \infty.$$
$$\therefore \quad \{a_n\} \quad \text{diverge.}$$

(g) $a_n = \dfrac{\ln n}{\ln(\ln n)}$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\ln n}{\ln(\ln n)} = \lim_{n\to\infty} \frac{(1/n)}{\left(\frac{1/n}{\ln n}\right)} = \lim_{n\to\infty} (\ln n) = \infty.$$

(h) $a_n = \dfrac{\int_0^n \ln(2+e^{-x})\, dx}{n}$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\int_0^n \ln(2+e^{-x})\, dx}{n} \stackrel{L}{=} \lim_{n\to\infty} \frac{\ln(2+e^{-n})}{1} = \ln\left(\lim_{n\to\infty} (2+e^{-n})\right) = \ln 2.$$

(i) $a_n = \dfrac{\int_0^{2n} e^{x^3}\,dx}{\int_0^{n} e^{8x^3}\,dx}$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\int_0^{2n} e^{x^3}\,dx}{\int_0^{n} e^{8x^3}\,dx} \overset{L}{=} \lim_{n\to\infty} \frac{e^{(2n)^3}2}{e^{8n^3}} = 2.$$

(j) $a_n = \sqrt{\dfrac{1+2n}{3+5n}}$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{\frac{1+2n}{3+5n}} = \sqrt{\lim_{n\to\infty} \frac{1+2n}{3+5n}} \overset{L}{=} \sqrt{\lim_{n\to\infty} \left(\frac{2}{5}\right)} = \sqrt{\frac{2}{5}}.$$

(k) $a_n = \ln n - \ln(n+1)$.

$$\begin{aligned} \lim_{n\to\infty} a_n &= \lim_{n\to\infty} [\ln n - \ln(n+1)] = \lim_{n\to\infty} \ln\left(\frac{n}{n+1}\right) \\ &= \ln\left[\lim_{n\to\infty}\left(\frac{n}{n+1}\right)\right] \overset{L}{=} \ln\left[\lim_{n\to\infty}\left(\frac{1}{1}\right)\right] = \ln 1 = 0. \end{aligned}$$

(l) $a_n = \dfrac{\ln(n^3)}{n}$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\ln(n^3)}{n} = \lim_{n\to\infty} \frac{3\ln n}{n} = 3\underbrace{\overbrace{\lim_{n\to\infty} \frac{\ln n}{n}}^{0}}_{\text{lím. frecuente}} = 0.$$

(m) $a_n = \dfrac{2^n - 1}{3^n}$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{2^n-1}{3^n}\right) = \lim_{n\to\infty} \left[\left(\frac{2}{3}\right)^n - \left(\frac{1}{3}\right)^n\right] = \underbrace{\overbrace{\lim_{n\to\infty}\left(\frac{2}{3}\right)^n}^{0} - \overbrace{\lim_{n\to\infty}\left(\frac{1}{3}\right)^n}^{0}}_{\text{lím. frecuentes}} = 0.$$

(n) $a_n = \dfrac{(-4)^n}{n!}$.

$$\lim_{n\to\infty} a_n = \underbrace{\overbrace{\lim_{n\to\infty} \frac{(-4)^n}{n!}}^{0}}_{\text{lím. frecuente}} = 0\ .$$

(o) $a_n = \left(\dfrac{3}{n}\right)^{1/n}$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{3}{n}\right)^{1/n} = \lim_{n\to\infty} \frac{3^{1/n}}{n^{1/n}} = \underbrace{\frac{\overbrace{\lim_{n\to\infty} 3^{1/n}}^{1}}{\underbrace{\lim_{n\to\infty} n^{1/n}}_{1}}}_{\text{lím. frecuentes}} = 1.$$

(p) $a_n = \left(1 - \frac{1}{n}\right)^n$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n = \underbrace{\lim_{n\to\infty} \left(1 + \frac{(-1)}{n}\right)^n}_{\text{lím. frecuente}} = e^{-1}.$$

(q) $a_n = n \ln \left(1 - \frac{1}{3n}\right)$.

$$\begin{aligned} \lim_{n\to\infty} a_n &= \lim_{n\to\infty} \left[n \ln \left(1 - \frac{1}{3n}\right)\right] = \lim_{n\to\infty} \ln \left(1 - \frac{1}{3n}\right)^n \\ &= \ln \left[\underbrace{\lim_{n\to\infty} \left(1 - \frac{1}{3n}\right)^n}_{\text{lím. frecuente}}\right] = \ln e^{-1/3} = -\frac{1}{3}. \end{aligned}$$

(r) $a_n = \left(\frac{n}{n+1}\right)^n$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \frac{1}{\left(\frac{n+1}{n}\right)^n} = \frac{1}{\underbrace{\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n}_{\text{lím. frecuente}}} = \frac{1}{e}.$$

(s) $a_n = (n+4)^{1/(n+4)}$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (n+4)^{1/(n+4)} = \lim_{n\to\infty} e^{\ln(n+4)^{1/(n+4)}} = e^{\lim\limits_{n\to\infty} \frac{\ln(n+4)}{n+4}} \overset{L}{=} e^{\lim\limits_{n\to\infty} \frac{1/(n+4)}{1}} = e^0 = 1.$$

(t) $a_n = n\,(1 - \cos(1/n))$.

$$\begin{aligned} \lim_{n\to\infty} a_n &= \lim_{n\to\infty} n\,(1 - \cos(1/n)) = \lim_{n\to\infty} \left[\frac{1 - \cos(1/n)}{1/n}\right] \overset{L}{=} \lim_{n\to\infty} \left[\frac{(-1/n^2)\ \mathit{sen}\,(1/n)}{(-1/n^2)}\right] \\ &= \lim_{n\to\infty} \mathit{sen}\,(1/n) = \mathit{sen}\,0 = 0. \end{aligned}$$

(u) $a_n = n\left(3^{1/n} - 1\right)$.

$$\begin{aligned} \lim_{n\to\infty} a_n &= \lim_{n\to\infty} \left[n\left(3^{1/n} - 1\right)\right] = \lim_{n\to\infty} \left[\frac{3^{1/n} - 1}{(1/n)}\right] \overset{L}{=} \lim_{n\to\infty} \left[\frac{3^{1/n}\,(\ln 3)\,(-1/n^2)}{(-1/n^2)}\right] \\ &= \underbrace{\lim_{n\to\infty} \left(3^{1/n} \ln 3\right)}_{\text{lím. frecuente}} = \ln 3. \end{aligned}$$

(v) $a_n = \left(1 - \frac{2}{n}\right)^{n/2}$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1 - \frac{2}{n}\right)^{n/2} \underbrace{=}_{\text{cambio de variable: } m=n/2} \overbrace{\lim_{m\to\infty} \left(1 - \frac{1}{m}\right)^m}^{\text{lím. frecuente}} = e^{-1}.$$

(w) $a_n = \dfrac{(2n)!}{n^2\,(2n-2)!}$.

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{(2n)!}{n^2\,(2n-2)!} = \lim_{n\to\infty} \frac{(2n)\,(2n-1)\,[(2n-2)!]}{n^2\,[(2n-2)!]} = \lim_{n\to\infty} \frac{4n^2-2n}{n^2} \\ &= \lim_{n\to\infty} \left(4-\frac{2}{n}\right) = 4.\end{aligned}$$

CÁLCULO III
TAREA 11 - SOLUCIONES
SERIES
(Temas 5.1-5.4)

1. (a) $\sum_{n=0}^{\infty} e^{-2n} = \sum_{n=0}^{\infty} (e^{-2})^n$ es una serie geométrica, con $r = e^{-2}$. Como $0 < e^{-2} < 1$, la serie converge. El valor de la suma es

$$\sum_{n=0}^{\infty} (e^{-2})^n = \frac{1}{1-e^{-2}} = \frac{e^2}{e^2-1}.$$

(b) $\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^n$ no es una serie geométrica (la base depende de n).

(c) $\sum_{n=1}^{\infty} 2^{2/n}$ no es una serie geométrica (la potencia no es entera).

(d) $\sum_{n=0}^{\infty} 2^{n/2} = \sum_{n=0}^{\infty} (\sqrt{2})^n$ es una serie geométrica, con $r = \sqrt{2}$. Como $\sqrt{2} > 1$, la serie diverge.

(e) $\sum_{n=0}^{\infty} (-1)^n (\ln 2)^n = \sum_{n=0}^{\infty} (-\ln 2)^n$ es una serie geométrica, con $r = -\ln 2$. Como $|r| < 1$, la serie converge. El valor de la suma es

$$\sum_{n=0}^{\infty} (-\ln 2)^n = \frac{1}{1-(-\ln 2)} = \frac{1}{1+\ln 2}.$$

(f) $\sum_{n=1}^{\infty} (\ln 2)^{1-n} = \sum_{n=1}^{\infty} \left(\frac{1}{\ln 2}\right)^{n-1}$ es una serie geométrica, con $r = \frac{1}{\ln 2}$. Como $r > 1$, la serie diverge.

(g) $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} = \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^{3/2}$ no es una serie geométrica (la base depende de n y la potencia es constante).

(h) $\sum_{n=1}^{\infty} \frac{r^n}{n!}$ no es una serie geométrica (le sobra $n!$ en el denominador).

2. (a)

$$\begin{aligned} 2 - \frac{1}{2} + \frac{1}{3} - \frac{2}{9} + \frac{4}{27} - \frac{8}{81} + \cdots &= 2 - \frac{1}{2} + \frac{1}{3}\left(1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \cdots\right) \\ &= 2 - \frac{1}{2} + \frac{1}{3}\left(1 + \left(-\frac{2}{3}\right) + \left(-\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right)^3 + \cdots\right) \\ &= 2 - \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{1-\left(-\frac{2}{3}\right)} = \frac{17}{10}. \end{aligned}$$

(b)

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^{n+2}\frac{5}{(-6)^n} &= \sum_{n=1}^{\infty}(-1)^3(-1)^{n-1}\frac{(5)}{(-6)(-6)^{n-1}} = \frac{(-1)^3(5)}{(-6)}\sum_{n=1}^{\infty}\left(\frac{1}{6}\right)^{n-1}\\ &= \frac{5}{6}\sum_{n=1}^{\infty}\left(\frac{1}{6}\right)^{n-1} = \frac{5}{6}\cdot\frac{1}{1-\left(\frac{1}{6}\right)} = 1.\end{aligned}$$

(c)

$$\begin{aligned}\sum_{n=2}^{\infty}(-1)^{n+2}\frac{(-6)}{3^n} &= \sum_{n=2}^{\infty}(-1)^4(-1)^{n-2}\frac{(-6)}{3^2\cdot 3^{n-2}} = \frac{(-6)(-1)^4}{9}\sum_{n=2}^{\infty}\frac{(-1)^{n-2}}{3^{n-2}}\\ &= -\frac{2}{3}\sum_{n=2}^{\infty}\left(-\frac{1}{3}\right)^{n-2} = -\frac{2}{3}\cdot\frac{1}{1-\left(-\frac{1}{3}\right)} = -\frac{2}{3}\left(\frac{3}{4}\right) = -\frac{1}{2}.\end{aligned}$$

(d)

$$\begin{aligned}\sum_{n=2}^{\infty}(-1)^{n+1}\,2^{2n}\,8^{1-n} &= \sum_{n=2}^{\infty}(-1)^{n+1}\left(2^2\right)^n\frac{1}{8^{n-1}} = \sum_{n=2}^{\infty}(-1)^{n+1}\frac{4^n}{8^{n-1}}\\ &= \sum_{n=2}^{\infty}(-1)^3(-1)^{n-2}\frac{4^2 4^{n-2}}{8\cdot 8^{n-2}} = \frac{(-1)^3 4^2}{8}\sum_{n=2}^{\infty}\left(-\frac{4}{8}\right)^{n-2}\\ &= -2\sum_{n=2}^{\infty}\left(-\frac{1}{2}\right)^{n-2} = -2\cdot\frac{1}{1-\left(-\frac{1}{2}\right)} = -\frac{4}{3}.\end{aligned}$$

(e)

$$\begin{aligned}ce^{-r}+ce^{-2r}+ce^{-3r}+ce^{-4r}+\cdots &= ce^{-r}\left(1+e^{-r}+e^{-2r}+e^{-3r}+\cdots\right)\\ &= ce^{-r}\left(1+e^{-r}+\left(e^{-r}\right)^2+\left(e^{-r}\right)^3+\cdots\right)\\ &= ce^{-r}\cdot\frac{1}{1-e^{-r}} = \frac{ce^{-r}}{1-e^{-r}}, \qquad r>0.\end{aligned}$$

(f)

$$\sum_{n=0}^{\infty}r(1-r)^n = r\sum_{n=0}^{\infty}(1-r)^n = r\cdot\frac{1}{1-(1-r)} = 1, \qquad 0<r<2.$$

(g)

$$\sum_{n=0}^{\infty}\frac{(1+g)^n D}{(1+r)^{n+1}} = \frac{D}{1+r}\sum_{n=0}^{\infty}\left(\frac{1+g}{1+r}\right)^n = \frac{D}{1+r}\cdot\frac{1}{1-\left(\frac{1+g}{1+r}\right)} = \frac{D}{r-g}, \qquad 0<g<r.$$

(h)

$$\begin{aligned}\sum_{k=1}^{\infty}(1+r)^{1-k}(1+r)^2\gamma^k &= \sum_{k=1}^{\infty}(1+r)^{3-k}\gamma^k = \sum_{k=1}^{\infty}\frac{\gamma^k}{(1+r)^{k-3}} = \sum_{k=1}^{\infty}\frac{\gamma\cdot\gamma^{k-1}}{(1+r)^{-2}(1+r)^{k-1}}\\ &= \gamma(1+r)^2\sum_{k=1}^{\infty}\left(\frac{\gamma}{1+r}\right)^{k-1} = \gamma(1+r)^2\cdot\frac{1}{1-\dfrac{\gamma}{1+r}}\\ &= \frac{\gamma(1+r)^3}{1+r-\gamma},\quad 0<\gamma<1+r,\ r>0\end{aligned}$$

3. (a) $\sum_{n=0}^{\infty}(-1)^n x^n = \sum_{n=0}^{\infty}(-x)^n$, de modo que $r=-x$.
La serie converge sólo si $|-x|=|x|<1$, es decir, si $-1<x<1$. En ese caso,

$$\sum_{n=0}^{\infty}(-1)^n x^n = \sum_{n=0}^{\infty}(-x)^n = \frac{1}{1-(-x)} = \frac{1}{1+x},\qquad -1<x<1.$$

(b) $\sum_{n=0}^{\infty}\frac{2^n}{x^n} = \sum_{n=0}^{\infty}\left(\frac{2}{x}\right)^n$, de modo que $r=\frac{2}{x}$.
La serie converge sólo si $\left|\frac{2}{x}\right| = \frac{2}{|x|}<1$, es decir, si $x<-2$ o $x>2$. En ese caso,

$$\sum_{n=0}^{\infty}\frac{2^n}{x^n} = \sum_{n=0}^{\infty}\left(\frac{2}{x}\right)^n = \frac{1}{1-\dfrac{2}{x}} = \frac{x}{x-2},\qquad x<-2\ \text{ o }\ x>2.$$

(c) $\sum_{n=0}^{\infty}e^{-nx^2} = \sum_{n=0}^{\infty}\left(e^{-x^2}\right)^n$, de modo que $r=e^{-x^2}$.
i) Si $x=0$, entonces $r=1$, de modo que la serie diverge.
ii) Si $x\neq 0$, entonces $0<r<1$, de modo que la serie converge. El valor de la suma es

$$\sum_{n=0}^{\infty}e^{-nx^2} = \sum_{n=0}^{\infty}\left(e^{-x^2}\right)^n = \frac{1}{1-e^{-x^2}} = \frac{e^{x^2}}{e^{x^2}-1},\qquad x\neq 0.$$

(d) $\sum_{n=0}^{\infty}(\ln x)^n$, de modo que $r=\ln x$.
La serie converge sólo si $|\ln x|<1$, es decir, si $e^{-1}<x<e$. En ese caso,

$$\sum_{n=0}^{\infty}(\ln x)^n = \frac{1}{1-\ln x},\qquad \frac{1}{e}<x<e.$$

4. (a) Partimos de la suma geométrica ($r\neq 1$, $k\geq 1$)

$$\sum_{n=0}^{k-1}r^n = \frac{1-r^k}{1-r}.$$

Esto es,

$$1+r+r^2+r^3+\cdots+r^{k-1}=\frac{1-r^k}{1-r}.$$

Derivando con respecto a r ambos lados de la igualdad, se tiene

$$\frac{d}{dr}\left(1+r+r^2+r^3+\cdots+r^{k-1}\right)=\frac{d}{dr}\left(\frac{1-r^k}{1-r}\right)$$

$$\therefore\ 1+2r+3r^2+\cdots+(k-1)\,r^{k-2}=\frac{1-kr^{k-1}+(k-1)r^k}{(1-r)^2}$$

Multiplicando por r ambos lados de la igualdad se llega a

$$r+2r^2+3r^3+\cdots+(k-1)\,r^{k-1}=\frac{r\left[1-kr^{k-1}+(k-1)r^k\right]}{(1-r)^2}$$

$$\therefore\ \sum_{n=0}^{k-1}nr^n=\frac{r\left[1-kr^{k-1}+(k-1)r^k\right]}{(1-r)^2}.$$

De esta manera,

$$\begin{aligned}\sum_{n=0}^{k-1}(a+nd)r^n &= a\sum_{n=0}^{k-1}r^n+d\sum_{n=0}^{k-1}nr^n\\ &= a\frac{(1-r^k)}{1-r}+\frac{rd\left[1-kr^{k-1}+(k-1)r^k\right]}{(1-r)^2}.\end{aligned}$$

(b) Como $|r|<1$, en el límite $k\to\infty$ se satisface

$$\lim_{k\to\infty}r^k=0 \quad \text{y} \quad \lim_{k\to\infty}kr^k=0.$$

El segundo resultado se obtiene usando la regla de L'Hopital (es del tipo $\infty\cdot 0$). Así,

$$\sum_{n=0}^{\infty}(a+nd)r^n=\frac{a}{1-r}+\frac{rd}{(1-r)^2}.$$

5. (a) Sea

$$S=\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)=\lim_{k\to\infty}\underbrace{\sum_{n=1}^{k}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)}_{S_k}=\lim_{k\to\infty}S_k.$$

Como

$$\begin{aligned}S_k &= \sum_{n=1}^{k}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)\\ &= \left(\frac{1}{1}-\frac{1}{\sqrt{2}}\right)+\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}}\right)+\ldots+\left(\frac{1}{\sqrt{k-1}}-\frac{1}{\sqrt{k}}\right)+\left(\frac{1}{\sqrt{k}}-\frac{1}{\sqrt{k+1}}\right)\\ &= 1-\frac{1}{\sqrt{k+1}},\end{aligned}$$

por lo tanto

$$S=\lim_{k\to\infty} S_k=\lim_{k\to\infty}\left(1-\frac{1}{\sqrt{k+1}}\right)=1.$$

De esta manera,

$$\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)=1.$$

(b) Sea

$$S=\sum_{n=-2}^{\infty}\left(e^{-n}-e^{1-n}\right)=\lim_{k\to\infty}\underbrace{\sum_{n=-2}^{k}\left(e^{-n}-e^{1-n}\right)}_{S_k}=\lim_{k\to\infty} S_k.$$

Como

$$\begin{aligned} S_k &= \sum_{n=-2}^{k}\left(e^{-n}-e^{1-n}\right)\\ &= \left(e^2-e^3\right)+\left(e^1-e^2\right)+\ldots+\left(e^{1-k}-e^{2-k}\right)+\left(e^{-k}-e^{1-k}\right)\\ &= -e^3+e^{-k}, \end{aligned}$$

por lo tanto

$$S=\lim_{k\to\infty} S_k=\lim_{k\to\infty}\left(-e^3+e^{-k}\right)=-e^3.$$

De esta manera,

$$\sum_{n=-2}^{\infty}\left(e^{-n}-e^{1-n}\right)=-e^3.$$

6. (a) $\sum\limits_{n=1}^{\infty}\frac{1}{2^{2/n}}$ diverge, ya que

$$\lim_{n\to\infty}\frac{1}{2^{2/n}}=\lim_{n\to\infty}\frac{1}{\left(2^{1/n}\right)\left(2^{1/n}\right)}=\frac{1}{\underbrace{\lim_{n\to\infty} 2^{1/n}}_{1}}\cdot\frac{1}{\underbrace{\lim_{n\to\infty} 2^{1/n}}_{1}}=1\neq 0.$$

(b) $\sum\limits_{n=1}^{\infty}(-1)^n\left(\frac{n-1}{n}\right)=\sum\limits_{n=1}^{\infty}(-1)^n\left(1-\frac{1}{n}\right)$ diverge, ya que

$$\lim_{n\to\infty}(-1)^n\left(1-\frac{1}{n}\right) \text{ no existe (hay dos puntos de acumulación, 1 y } -1).$$

(c) $\sum\limits_{n=1}^{\infty}\left(1-\frac{3}{n}\right)^n$ diverge, ya que

$$\lim_{n\to\infty}\left(1-\frac{3}{n}\right)^n=e^{-3}\neq 0.$$

(d) $\sum_{n=0}^{\infty} \frac{n!}{1000^n}$ diverge, ya que

$$\lim_{n\to\infty} \frac{n!}{1000^n} = \frac{1}{\underbrace{\lim_{n\to\infty} \frac{1000^n}{n!}}_{0}} \quad \text{no existe.}$$

(e) $\sum_{n=0}^{\infty} \cos(n\pi)$ diverge, ya que

$$\lim_{n\to\infty} \cos(n\pi) \quad \text{no existe.}$$

7. (a) $\sum_{n=1}^{\infty} \frac{(\ln n)^n}{n^n}$.

$$a_n = \frac{(\ln n)^n}{n^n} = \left(\frac{\ln n}{n}\right)^n$$

$$\therefore \quad \sqrt[n]{a_n} = \sqrt[n]{\left(\frac{\ln n}{n}\right)^n} = \frac{\ln n}{n}$$

$$\therefore \quad \lim_{n\to\infty} \sqrt[n]{a_n} = \underbrace{\lim_{n\to\infty} \frac{\ln n}{n}}_{0} = 0 < 1$$

$$\therefore \quad \sum_{n=1}^{\infty} \frac{(\ln n)^n}{n^n} \quad \text{converge.}$$

(b) $\sum_{n=1}^{\infty} n^2 e^{-n}$.

$$a_n = n^2 e^{-n} = \frac{n^2}{e^n}$$

$$\therefore \quad \sqrt[n]{a_n} = \sqrt[n]{\frac{n^2}{e^n}} = \frac{1}{e} \sqrt[n]{n}\,\sqrt[n]{n}$$

$$\therefore \quad \lim_{n\to\infty} \sqrt[n]{a_n} = \frac{1}{e} \underbrace{\lim_{n\to\infty} n^{1/n}}_{1} \cdot \underbrace{\lim_{n\to\infty} n^{1/n}}_{1} = \frac{1}{e} < 1$$

$$\therefore \quad \sum_{n=1}^{\infty} n^2 e^{-n} \quad \text{converge.}$$

(c) $\sum_{n=1}^{\infty} \frac{n!}{10^n}$.

$$a_n = \frac{n!}{10^n}, \qquad a_{n+1} = \frac{(n+1)!}{10^{n+1}}$$

$$\therefore \quad \frac{a_{n+1}}{a_n} = \frac{(n+1)!\,10^n}{10^{n+1}\,n!} = \frac{(n+1)!}{10\,n!} = \frac{(n+1)\,n!}{10\,n!} = \frac{(n+1)}{10}$$

$$\therefore \quad \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{10} \lim_{n\to\infty} (n+1) = \infty$$

$$\therefore \quad \sum_{n=1}^{\infty} \frac{n!}{10^n} \quad \text{diverge.}$$

(d) $\sum_{n=1}^{\infty}\left(\frac{n+1}{n}\right)^n$.

$$a_n=\left(\frac{n+1}{n}\right)^n=\left(1+\frac{1}{n}\right)^n$$

$$\therefore\quad \sqrt[n]{a_n}=\sqrt[n]{\left(1+\frac{1}{n}\right)^n}=1+\frac{1}{n}$$

$$\therefore\quad \lim_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)=1\qquad\therefore \text{ la prueba no es concluyente.}$$

Utilizamos la prueba del $n-$ésimo término:

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e\neq 0$$

$$\therefore\quad \sum_{n=1}^{\infty}\left(\frac{n+1}{n}\right)^n \text{ diverge.}$$

(e) $\sum_{n=1}^{\infty}\frac{n^n}{2^{n^2}}$.

$$a_n=\frac{n^n}{2^{n^2}}$$

$$\therefore\quad \sqrt[n]{a_n}=\sqrt[n]{\frac{n^n}{2^{n^2}}}=\frac{n}{2^{n^2/n}}=\frac{n}{2^n}$$

$$\therefore\quad \lim_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\frac{n}{2^n}\overset{L}{=}\lim_{n\to\infty}\frac{1}{2^n\ln 2}=0<1$$

$$\therefore\quad \sum_{n=1}^{\infty}\frac{n^n}{2^{n^2}} \text{ converge.}$$

(f) $\sum_{n=2}^{\infty}\frac{n}{(\ln n)^n}$.

$$a_n=\frac{n}{(\ln n)^n}$$

$$\therefore\quad \sqrt[n]{a_n}=\sqrt[n]{\frac{n}{(\ln n)^n}}=\frac{n^{1/n}}{\ln n}$$

$$\therefore\quad \lim_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\frac{n^{1/n}}{\ln n}=\underbrace{\lim_{n\to\infty}n^{1/n}}_{1}\cdot\underbrace{\lim_{n\to\infty}\frac{1}{\ln n}}_{0}=0<1$$

$$\therefore\quad \sum_{n=2}^{\infty}\frac{n}{(\ln n)^n} \text{ converge.}$$

(g) $\sum_{n=1}^{\infty}\frac{3^n}{n^3 2^n}$.

$$a_n=\frac{3^n}{n^3 2^n},\qquad a_{n+1}=\frac{3^{n+1}}{(n+1)^3\,2^{n+1}}$$

$$\therefore\quad \frac{a_{n+1}}{a_n}=\frac{3^{n+1}\;n^3\;2^n}{(n+1)^3\;2^{n+1}\;3^n}=\frac{3n^3}{2\,(n+1)^3}$$

$$\therefore\quad \lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{3n^3}{2\,(n+1)^3}\overset{L}{=}\frac{3}{2}>1$$

$$\therefore \quad \sum_{n=1}^{\infty} \frac{3^n}{n^3 2^n} \quad \text{diverge.}$$

(h) $\displaystyle\sum_{n=1}^{\infty} \frac{(n+3)!}{3!\; n!\; 3^n}$.

$$a_n = \frac{(n+3)!}{3!\; n!\; 3^n}$$

$$\therefore \quad a_{n+1} = \frac{((n+1)+3)!}{3!\;\; (n+1)!\; 3^{n+1}} = \frac{(n+4)!}{3!\;\; (n+1)!\; 3^{n+1}}$$

$$\therefore \quad \frac{a_{n+1}}{a_n} = \frac{(n+4)!\; 3!\; n!\; 3^n}{3!\;\; (n+1)!\; 3^{n+1}\;\; (n+3)!} = \frac{(n+4)!\; n!}{3\;\; (n+1)!\;\; (n+3)!}$$

$$= \frac{(n+4)\;\; (n+3)!\; n!}{3\;\; (n+1)\;\; n!\;\; (n+3)!} = \frac{(n+4)}{3\,(n+1)}$$

$$\therefore \quad \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{3} \lim_{n\to\infty} \frac{n+4}{n+1} = \frac{1}{3} < 1$$

$$\therefore \quad \sum_{n=1}^{\infty} \frac{(n+3)!}{3!\; n!\; 3^n} \quad \text{converge.}$$

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$.

$$a_n = \frac{n^n}{n!}, \qquad a_{n+1} = \frac{(n+1)^{n+1}}{(n+1)!}$$

$$\therefore \quad \frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}\;\; n!}{(n+1)!\; n^n} = \frac{(n+1)\;\; (n+1)^n\;\; n!}{(n+1)\;\; n!\; n^n} = \frac{(n+1)^n}{n^n} = \left(\frac{n+1}{n}\right)^n$$

$$\therefore \quad \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e > 1$$

$$\therefore \quad \sum_{n=1}^{\infty} \frac{n^n}{n!} \quad \text{diverge.}$$

(j) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(2n+1)!}$.

$$a_n = \frac{n!}{(2n+1)!}$$

$$\therefore \quad a_{n+1} = \frac{(n+1)!}{(2\,(n+1)+1)!} = \frac{(n+1)!}{(2n+3)!}$$

$$\therefore \quad \frac{a_{n+1}}{a_n} = \frac{(n+1)!\;\; (2n+1)!}{(2n+3)!\; n!} = \frac{(n+1)\; n!\;\; (2n+1)!}{(2n+3)\;\; (2n+2)\;\; (2n+1)!\; n!} = \frac{n+1}{(2n+3)\;\; (2n+2)}$$

$$\therefore \quad \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+1}{(2n+3)\;\; 2\;\; (n+1)} = \lim_{n\to\infty} \frac{1}{4n+6} = 0 < 1$$

$$\therefore \quad \sum_{n=1}^{\infty} \frac{n!}{(2n+1)!} \quad \text{converge.}$$

8. (a) $\displaystyle\sum_{n=1}^{\infty} 2^{-n}$.

Sabemos que

$$\int_1^{\infty} 2^{-x}\, dx = \lim_{b\to\infty} \int_1^b 2^{-x}\, dx = \lim_{b\to\infty} \left[-\frac{2^{-x}}{\ln 2}\right]_1^b = \lim_{b\to\infty} \left(-\frac{1}{2^b \ln 2} + \frac{1}{2\ln 2}\right) = \frac{1}{2\ln 2}.$$

$$\therefore \quad \int_1^\infty 2^{-x}\,dx \quad \text{converge}$$

$$\therefore \quad \sum_{n=1}^\infty 2^{-n} \quad \text{converge.}$$

Este resultado establece que la serie dada converge, pero no dice a qué valor. De hecho, es fácil mostrar que $\sum\limits_{n=1}^\infty 2^{-n} = 1$ (la serie es tipo geométrica, con $r = 1/2$).

Observa que $\sum\limits_{n=1}^\infty 2^{-n} \neq \int_1^\infty 2^{-x}\,dx$.

(b) $\sum\limits_{n=1}^\infty \dfrac{1}{2n+1}$.

Sabemos que

$$\int_1^\infty \frac{1}{2x+1}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{2x+1}\,dx = \frac{1}{2}\lim_{b\to\infty}\left[\ln|2x+1|\right]_1^b = \frac{1}{2}\lim_{b\to\infty}\left(\ln(2b+1) - \ln 3\right).$$

$$\therefore \quad \int_1^\infty \frac{1}{2x+1}\,dx \quad \text{diverge}$$

$$\therefore \quad \sum_{n=1}^\infty \frac{1}{2n+1} \quad \text{diverge.}$$

(c) $\sum\limits_{n=2}^\infty \dfrac{1}{n(\ln n)^2}$.

Sabemos que

$$\int_2^\infty \frac{1}{x(\ln x)^2}\,dx = \lim_{b\to\infty}\int_2^b \frac{1}{x(\ln x)^2}\,dx = \lim_{b\to\infty}\left[\frac{1}{\ln x}\right]_2^b = \frac{1}{\ln 2}.$$

$$\therefore \quad \int_2^\infty \frac{1}{x(\ln x)^2}\,dx \quad \text{converge}$$

$$\therefore \quad \sum_{n=2}^\infty \frac{1}{n(\ln n)^2} \quad \text{converge.}$$

(d) $\sum\limits_{n=1}^\infty \dfrac{1}{\sqrt{n}\left(\sqrt{n}+1\right)^2}$.

Sabemos que

$$\int_1^\infty \frac{dx}{\sqrt{x}\left(\sqrt{x}+1\right)^2} = \lim_{b\to\infty}\int_1^b \frac{dx}{\sqrt{x}\left(\sqrt{x}+1\right)^2} = \lim_{b\to\infty}\left[-\frac{2}{\sqrt{x}+1}\right]_1^b = 1.$$

$$\therefore \quad \int_1^\infty \frac{dx}{\sqrt{x}\left(\sqrt{x}+1\right)^2} \quad \text{converge}$$

$$\therefore \quad \sum_{n=1}^\infty \frac{1}{\sqrt{n}\left(\sqrt{n}+1\right)^2} \quad \text{converge.}$$

9. (a) $\sum_{n=1}^{\infty} \frac{1}{3^n+1}$.

Como $\frac{1}{3^n+1} < \frac{1}{3^n}$, por lo tanto $\sum_{n=1}^{\infty} \left(\frac{1}{3^n+1}\right) < \sum_{n=1}^{\infty} \frac{1}{3^n}$.

Sabemos que $\sum_{n=1}^{\infty} \frac{1}{3^n}$ converge (tipo geométrica, con $r = 1/3$).

Por lo tanto $\sum_{n=1}^{\infty} \left(\frac{1}{3^n+1}\right)$ converge.

(b) $\sum_{n=1}^{\infty} \frac{3}{n+\sqrt{n}}$.

Sabemos que $\frac{3}{n+\sqrt{n}} \geq \frac{3}{n+n} = \frac{3}{2n} > \frac{1}{n}$.

Como $\frac{3}{n+\sqrt{n}} > \frac{1}{n}$, por lo tanto $\sum_{n=1}^{\infty} \frac{3}{n+\sqrt{n}} > \sum_{n=1}^{\infty} \frac{1}{n}$.

Sabemos que $\sum_{n=1}^{\infty} \frac{1}{n}$ diverge (serie armónica, o serie-p con $p = 1$).

Por lo tanto $\sum_{n=1}^{\infty} \frac{3}{n+\sqrt{n}}$ diverge.

(c) $\sum_{n=1}^{\infty} \frac{n+1}{n^{5/2}}$.

Sabemos que $\frac{n+1}{n^{5/2}} \leq \frac{n+n}{n^{5/2}} = \frac{2}{n^{3/2}} < \frac{3}{n^{3/2}}$.

Como $\frac{n+1}{n^{5/2}} < \frac{3}{n^{3/2}}$, por lo tanto $\sum_{n=1}^{\infty} \frac{n+1}{n^{5/2}} < 3\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$.

Sabemos que $3\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converge (serie-p con $p = 3/2$).

Por lo tanto $\sum_{n=1}^{\infty} \frac{n+1}{n^{5/2}}$ converge.

(d) $\sum_{n=1}^{\infty} \frac{1}{n2^n}$.

Sabemos que $\frac{1}{n2^n} \leq \frac{1}{2^n} < \frac{2}{2^n} = \frac{1}{2^{n-1}}$.

Como $\frac{1}{n2^n} < \frac{1}{2^{n-1}}$, por lo tanto $\sum_{n=1}^{\infty} \frac{1}{n2^n} < \sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$.

Sabemos que $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ converge (geométrica, con $r = 1/2$).

Por lo tanto $\sum_{n=1}^{\infty} \frac{1}{n2^n}$ converge.

(e) $\sum_{n=1}^{\infty} \left(\frac{n}{3n+1}\right)^n$.

Sabemos que $\left(\frac{n}{3n+1}\right)^n < \left(\frac{n}{3n}\right)^n = \left(\frac{1}{3}\right)^n$.

Como $\left(\frac{n}{3n+1}\right)^n < \left(\frac{1}{3}\right)^n$, por lo tanto $\sum_{n=1}^{\infty} \left(\frac{n}{3n+1}\right)^n < \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$.

Sabemos que $\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n$ converge (tipo geométrica, con $r = 1/3$).

Por lo tanto $\sum_{n=1}^{\infty}\left(\frac{n}{3n+1}\right)^n$ converge.

(f) $\sum_{n=3}^{\infty}\frac{1}{\ln(\ln n)}$.

Sabemos que $\ln(\ln n) < \ln n < n$, para $n \geq 3$.

Como $\frac{1}{\ln(\ln n)} > \frac{1}{n}$, por lo tanto $\sum_{n=3}^{\infty}\frac{1}{\ln(\ln n)} > \sum_{n=3}^{\infty}\frac{1}{n}$.

Sabemos que $\sum_{n=3}^{\infty}\frac{1}{n}$ diverge (tipo armónica, o serie-p con $p = 1$).

Por lo tanto $\sum_{n=3}^{\infty}\frac{1}{\ln(\ln n)}$ diverge.

10. (a) $\sum_{n=1}^{\infty}\frac{1}{3^n-1}$.

Sean $a_n = \frac{1}{3^n-1}$, $b_n = \frac{1}{3^n}$. Sabemos que $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n$ converge (tipo geométrica, con $r = 1/3$) y que

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{3^n}{3^n-1} \overset{L}{=} \lim_{n\to\infty}\frac{3^n\ln 3}{3^n\ln 3} = 1.$$

$\therefore \sum_{n=1}^{\infty}\frac{1}{3^n-1}$ converge.

(b) $\sum_{n=1}^{\infty}\frac{10n+1}{n(n+1)(n+2)}$.

Sean $a_n = \frac{10n+1}{n(n+1)(n+2)}$, $b_n = \frac{1}{n^2}$. Sabemos que $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{1}{n^2}$ converge (serie-p, con $p = 2 > 1$) y que

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{(10n+1)\,n^2}{n(n+1)(n+2)} = \lim_{n\to\infty}\frac{10n^3+n^2}{n^3+3n^2+2n} \overset{L}{=} 10.$$

$\therefore \sum_{n=1}^{\infty}\frac{10n+1}{n(n+1)(n+2)}$ converge.

(c) $\sum_{n=2}^{\infty}\frac{1}{\sqrt{n}\,\ln n}$.

Sean $a_n = \frac{1}{\sqrt{n}\,\ln n}$, $b_n = \frac{1}{n}$. Sabemos que $\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty}\frac{1}{n}$ diverge (tipo armónica, o serie-p con $p = 1$) y que

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n}{\sqrt{n}\ln n} = \lim_{n\to\infty}\frac{\sqrt{n}}{\ln n} \overset{L}{=} \lim_{n\to\infty}\frac{1/(2\sqrt{n})}{1/n} = \lim_{n\to\infty}\frac{\sqrt{n}}{2} = \infty.$$

$\therefore \sum_{n=2}^{\infty}\frac{1}{\sqrt{n}\,\ln n}$ diverge.

(d) $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^3}$.

Sean $a_n = \frac{(\ln n)^2}{n^3}$, $b_n = \frac{1}{n^2}$. Sabemos que $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converge (serie-p con $p = 2 > 1$) y que

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(\ln n)^2}{n} \overset{L}{=} \lim_{n\to\infty} \frac{2(\ln n)(1/n)}{1} = 2 \lim_{n\to\infty} \frac{\ln n}{n} = 0.$$

$\therefore \quad \sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^3}$ converge.

11. (a) $1 - \frac{1}{5} + \frac{1}{25} - \frac{1}{125} + \frac{1}{625} - \cdots = \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{5}\right)^n$.

$\sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^n$ converge (geométrica con $r = 1/5 < 1$).

$\sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{5}\right)^n$ converge, ya que $\sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^n$ converge.

$\therefore \quad \sum_{n=0}^{\infty} (-5)^{-n}$ converge absolutamente.

(b) $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} - \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n}}$.

$\sum_{n=1}^{\infty} \left|(-1)^{n+1} \frac{1}{\sqrt{n}}\right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverge (serie-p con $p = 1/2 \leq 1$).

$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n}}$ converge (serie alternante, $a_n = \frac{1}{\sqrt{n}}$, $a_{n+1} < a_n$, $\lim_{n\to\infty} a_n = 0$).

$\therefore \quad \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n}}$ converge condicionalmente.

(c) $\sum_{n=1}^{\infty} (-1)^n \frac{n!}{2^n}$.

Prueba del n-ésimo término:

$\lim_{n\to\infty} \left[(-1)^n \frac{n!}{2^n}\right] \neq 0$.

$\therefore \quad \sum_{n=1}^{\infty} (-1)^n \frac{n!}{2^n}$ diverge.

(d) $\sum_{n=3}^{\infty} (-1)^{n+1} \frac{1}{\ln n}$.

$\sum_{n=3}^{\infty} \frac{1}{\ln n}$ diverge (ya que $\sum_{n=3}^{\infty} \frac{1}{\ln n} > \sum_{n=3}^{\infty} \frac{1}{n}$ y $\sum_{n=3}^{\infty} \frac{1}{n}$ diverge).

$\sum_{n=3}^{\infty} (-1)^{n+1} \frac{1}{\ln n}$ converge (serie alternante, $a_n = \frac{1}{\ln n}$, $a_{n+1} < a_n$, $\lim_{n\to\infty} a_n = 0$).

$\therefore \quad \sum_{n=3}^{\infty} (-1)^{n+1} \frac{1}{\ln n}$ converge condicionalmente.

(e) $\sum_{n=1}^{\infty} (-1)^n \dfrac{n}{n+1}$.

Prueba del n-ésimo término:

$$\lim_{n\to\infty} \left[(-1)^n \frac{n}{n+1}\right] \neq 0 \quad \text{(hay dos puntos de acumulación, 1 y } -1\text{)}.$$

$$\therefore \quad \sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1} \quad \text{diverge.}$$

(f) $\sum_{n=2}^{\infty} (-1)^n \left(\dfrac{\ln n}{\ln n^2}\right)^n$.

$$\sum_{n=2}^{\infty} \left(\frac{\ln n}{\ln n^2}\right)^n = \sum_{n=2}^{\infty} \left(\frac{\ln n}{2\ln n}\right)^n = \sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^n \quad \text{converge (tipo geométrica con } r = 1/2 < 1\text{).}$$

$$\sum_{n=2}^{\infty} (-1)^n \left(\frac{\ln n}{\ln n^2}\right)^n \quad \text{converge, ya que } \sum_{n=2}^{\infty} \left(\frac{\ln n}{\ln n^2}\right)^n \text{ converge.}$$

$$\therefore \quad \sum_{n=2}^{\infty} (-1)^n \left(\frac{\ln n}{\ln n^2}\right)^n \quad \text{converge absolutamente.}$$

(g) $\sum_{n=1}^{\infty} \dfrac{(-2)^{n+1}}{n+5^n}$.

$$\sum_{n=1}^{\infty} \frac{2^{n+1}}{n+5^n} \quad \text{converge (prueba del cociente: } \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{2}{5} < 1\text{).}$$

$$\sum_{n=1}^{\infty} \frac{(-2)^{n+1}}{n+5^n} \quad \text{converge, ya que } \sum_{n=1}^{\infty} \frac{2^{n+1}}{n+5^n} \text{ converge.}$$

$$\therefore \quad \sum_{n=1}^{\infty} \frac{(-2)^{n+1}}{n+5^n} \quad \text{converge absolutamente.}$$

12. (a) $\sum_{n=0}^{\infty} \dfrac{n}{n+1}$.

Prueba del n-ésimo término:

$$\lim_{n\to\infty} \frac{n}{n+1} \overset{L}{=} \lim_{n\to\infty} \frac{1}{1} = 1 \neq 0.$$

$$\therefore \quad \sum_{n=0}^{\infty} \frac{n}{n+1} \quad \text{diverge.}$$

(b) $\sum_{n=1}^{\infty} \dfrac{2}{n\sqrt{n}}$.

$$\sum_{n=1}^{\infty} \frac{2}{n\sqrt{n}} = 2\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \quad \text{serie-}p \text{ con } p = 3/2 > 1.$$

$$\therefore \quad \sum_{n=1}^{\infty} \frac{2}{n\sqrt{n}} \quad \text{converge.}$$

(c) $\sum_{n=1}^{\infty} \dfrac{1+\cos n}{n^3}$.

Prueba de la comparación directa:

$$\sum_{n=1}^{\infty} \frac{1+\cos n}{n^3} < \sum_{n=1}^{\infty} \frac{2}{n^3} \quad \text{y} \quad \sum_{n=1}^{\infty} \frac{2}{n^3} \text{ converge (serie } p \text{ con } p = 3 > 1\text{).}$$

$$\therefore \quad \sum_{n=1}^{\infty} \frac{1+\cos n}{n^2} \quad \text{converge.}$$

(d) $\sum_{n=0}^{\infty} e^{-n\pi}$.

$$\sum_{n=0}^{\infty} e^{-n\pi} = \sum_{n=0}^{\infty} (e^{-\pi})^n \quad \text{serie geométrica con } r = e^{-\pi} < 1.$$

$$\therefore \quad \sum_{n=0}^{\infty} e^{-n\pi} \quad \text{converge.}$$

(e) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$.

Prueba del cociente:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \frac{1}{\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1.$$

$$\therefore \quad \sum_{n=1}^{\infty} \frac{n!}{n^n} \quad \text{converge.}$$

(f) $\sum_{n=0}^{\infty} \frac{1}{2^{-n}}$.

$$\sum_{n=0}^{\infty} \frac{1}{2^{-n}} = \sum_{n=0}^{\infty} 2^n \quad \text{serie geométrica con } r = 2 > 1.$$

$$\therefore \quad \sum_{n=0}^{\infty} \frac{1}{2^{-n}} \quad \text{diverge.}$$

(g) $\sum_{n=1}^{\infty} n^{-1/2}$.

$$\sum_{n=1}^{\infty} n^{-1/2} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \quad \text{serie-}p \text{ con } p = 1/2 \le 1.$$

$$\therefore \quad \sum_{n=1}^{\infty} n^{-1/2} \quad \text{diverge.}$$

(h) $\sum_{n=3}^{\infty} \frac{\ln n}{n}$.

Prueba de la integral:

$$\int_3^{\infty} \frac{\ln x}{x} dx = \lim_{b\to\infty} \int_3^b \frac{\ln x}{x} dx = \lim_{b\to\infty} \left[\frac{(\ln x)^2}{2}\right]_3^b \qquad \text{diverge}$$

$$\therefore \quad \sum_{n=3}^{\infty} \frac{\ln n}{n} \quad \text{diverge.}$$

También puede utilizarse la prueba de la comparación directa:

$$\sum_{n=3}^{\infty} \frac{\ln n}{n} > \sum_{n=3}^{\infty} \frac{1}{n} \text{ y } \sum_{n=3}^{\infty} \frac{1}{n} \text{ diverge (tipo armónica).}$$

$$\therefore \quad \sum_{n=3}^{\infty} \frac{\ln n}{n} \quad \text{diverge.}$$

(i) $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{\ln n}$.

Prueba del n-ésimo término:

$$\lim_{n\to\infty} \frac{\sqrt{n}}{\ln n} \overset{L}{=} \lim_{n\to\infty} \frac{\frac{1}{2\sqrt{n}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{2} = \infty \neq 0.$$

$\therefore \quad \sum_{n=2}^{\infty} \frac{\sqrt{n}}{\ln n}$ diverge.

(j) $\sum_{n=1}^{\infty} \frac{2^{2n}}{n^n}$.

Prueba de la raíz n-ésima:

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{2^{2n}}{n^n}} = \lim_{n\to\infty} \frac{2^2}{n} = 0 < 1.$$

$\therefore \quad \sum_{n=1}^{\infty} \frac{2^{2n}}{n^n}$ converge.

(k) $\sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{\ln n}$.

Serie alternante: $a_n = \frac{1}{\ln n}$, $a_{n+1} < a_n$, $\lim_{n\to\infty} a_n = 0$.

$\therefore \quad \sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{\ln n}$ converge.

(l) $\sum_{n=0}^{\infty} ne^{-n^2}$

Prueba de la integral:

$$\int_0^{\infty} xe^{-x^2}dx = \lim_{b\to\infty} \int_0^b xe^{-x^2}dx = \lim_{b\to\infty} \left[-\tfrac{1}{2}e^{-x^2}\right]_0^b = \tfrac{1}{2} \quad \text{converge.}$$

$\therefore \quad \sum_{n=0}^{\infty} ne^{-n^2}$ converge.

13. (a) $\sum_{n=0}^{\infty} \frac{nx^n}{n+2}$.

$$\sum_{n=0}^{\infty} \frac{nx^n}{n+2} = \frac{x}{3} + \frac{2x^2}{4} + \frac{3x^3}{5} + \frac{4x^4}{6} + \frac{5x^5}{7} + \ldots$$

$$\therefore \quad a_n = \frac{nx^n}{n+2}$$

$$\therefore \quad a_{n+1} = \frac{(n+1)x^{n+1}}{(n+1)+2} = \frac{(n+1)x^{n+1}}{n+3}$$

$$\therefore \quad \frac{a_{n+1}}{a_n} = \frac{(n+1)\,x^{n+1}\,(n+2)}{(n+3)\,nx^n} = \frac{x\,(n+1)\,(n+2)}{(n+3)\,n}$$

$$\therefore \quad \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x\,(n+1)\,(n+2)}{(n+3)\,n}\right| = |x| \left(\frac{n^2+3n+2}{n^2+3n}\right)$$

$$\therefore \quad \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x| \underbrace{\lim_{n\to\infty}\left(\frac{n^2+3n+2}{n^2+3n}\right)}_{1} = |x| < 1$$

$\therefore \quad -1 < x < 1.$

En $x=1$: $\displaystyle\sum_{n=0}^{\infty}\frac{n(1)^n}{n+2} = \sum_{n=0}^{\infty}\frac{n}{n+2}$ diverge (prueba del n-ésimo término).

En $x=-1$: $\displaystyle\sum_{n=0}^{\infty}\frac{n(-1)^n}{n+2} = \sum_{n=0}^{\infty}(-1)^n\frac{n}{n+2}$ diverge (prueba del n-ésimo término).

$\therefore$ La serie converge en el intervalo $-1 < x < 1$ y el radio es $R = 1$.

(b) $\displaystyle\sum_{n=0}^{\infty}\frac{n(x+3)^n}{5^n}$.

$$\sum_{n=0}^{\infty}\frac{n(x+3)^n}{5^n} = \frac{x+3}{5} + \frac{2(x+3)^2}{5^2} + \frac{3(x+3)^3}{5^3} + \frac{4(x+3)^4}{5^4} + \ldots$$

$$\therefore \quad a_n = \frac{n(x+3)^n}{5^n}$$

$$\therefore \quad a_{n+1} = \frac{(n+1)(x+3)^{n+1}}{5^{n+1}}$$

$$\therefore \quad \frac{a_{n+1}}{a_n} = \frac{(n+1)(x+3)^{n+1}5^n}{5^{n+1}n(x+3)^n} = \frac{(n+1)(x+3)}{5n}$$

$$\therefore \quad \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)(x+3)}{5n}\right| = \frac{1}{5}|x+3|\left(\frac{n+1}{n}\right)$$

$$\therefore \quad \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{5}|x+3| \underbrace{\lim_{n\to\infty}\left(\frac{n+1}{n}\right)}_{1} = \frac{1}{5}|x+3| < 1$$

$\therefore \quad |x+3| < 5 \qquad \therefore \quad -8 < x < 2.$

En $x=-8$: $\displaystyle\sum_{n=0}^{\infty}\frac{n(-5)^n}{5^n} = \sum_{n=0}^{\infty}(-1)^n n = -1+2-3+4-\ldots$ diverge

En $x=2$: $\displaystyle\sum_{n=0}^{\infty}\frac{n(5)^n}{5^n} = \sum_{n=0}^{\infty}n = 1+2+3+4+\ldots$ diverge

$\therefore$ La serie converge en el intervalo $-8 < x < 2$ y el radio es $R = 5$.

(c) $\displaystyle\sum_{n=1}^{\infty}\frac{(x-1)^n}{2^n\sqrt{n}}$.

$$\sum_{n=1}^{\infty}\frac{(x-1)^n}{2^n\sqrt{n}} = \frac{(x-1)}{2\sqrt{1}} + \frac{(x-1)^2}{2^2\sqrt{2}} + \frac{(x-1)^3}{2^3\sqrt{3}} + \ldots$$

$$\therefore \quad a_n = \frac{(x-1)^n}{2^n\sqrt{n}}$$

$$\therefore \quad a_{n+1} = \frac{(x-1)^{n+1}}{2^{n+1}\sqrt{n+1}}$$

$$\therefore \quad \frac{a_{n+1}}{a_n} = \frac{(x-1)^{n+1}2^n\sqrt{n}}{2^{n+1}\sqrt{n+1}(x-1)^n} = \frac{(x-1)\sqrt{n}}{2\sqrt{n+1}}$$

$$\therefore \quad \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-1)\sqrt{n}}{2\sqrt{n+1}}\right| = \frac{1}{2}|x-1|\sqrt{\frac{n}{n+1}}$$

$$\therefore \quad \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{2}|x-1| \lim_{n\to\infty}\sqrt{\frac{n}{n+1}} = \frac{1}{2}|x-1| \underbrace{\sqrt{\lim_{n\to\infty}\frac{n}{n+1}}}_{1} = \frac{1}{2}|x-1| < 1$$

$$\therefore \quad |x-1| < 2 \qquad \therefore \quad -1 < x < 3.$$

En $x=-1$: $\displaystyle\sum_{n=1}^{\infty}\frac{(-2)^n}{2^n\sqrt{n}} = \sum_{n=1}^{\infty}\frac{(-1)^n 2^n}{2^n\sqrt{n}} = \sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt{n}}$ converge (serie alternante).

En $x=3$: $\displaystyle\sum_{n=1}^{\infty}\frac{(2)^n}{2^n\sqrt{n}} = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverge (serie-p con $p=1/2<1$).

$\therefore$ La serie converge en el intervalo $-1 \leq x < 3$ y el radio es $R=2$.

(d) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x+2)^n}{n2^n}$.

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x+2)^n}{n2^n} = \frac{(x+2)}{1\cdot 2} - \frac{(x+2)^2}{1\cdot 2^2} + \frac{(x+2)^3}{3\cdot 2^3} - \frac{(x+2)^4}{4\cdot 2^4} + ...$$

$$\therefore \quad a_n = \frac{(-1)^{n+1}(x+2)^n}{n2^n}$$

$$\therefore \quad |a_n| = \left|\frac{(-1)^{n+1}(x+2)^n}{n2^n}\right| = \frac{|x+2|^n}{n2^n}$$

$$\therefore \quad \sqrt[n]{|a_n|} = \sqrt[n]{\frac{|x+2|^n}{n2^n}} = \frac{|x+2|}{2\,n^{1/n}}$$

$$\therefore \quad \lim_{n\to\infty}\sqrt[n]{|a_n|} = \frac{|x+2|}{2}\lim_{n\to\infty}\frac{1}{n^{1/n}} = \frac{|x+2|}{2}\frac{1}{\underbrace{\lim_{n\to\infty} n^{1/n}}_{1}} = \frac{|x+2|}{2} < 1.$$

$$\therefore \quad |x+2| < 2 \qquad \therefore \quad -4 < x < 0.$$

En $x=-4$: $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(-2)^n}{n2^n} = -\sum_{n=1}^{\infty}\frac{1}{n}$ diverge (armónica).

En $x=0$: $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(2)^n}{n2^n} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ converge (armónica alternante).

$\therefore$ La serie converge en el intervalo $-4 < x \leq 0$ y el radio es $R=2$.

(e) $\displaystyle\sum_{n=0}^{\infty}\frac{3^n x^n}{n!}$.

$$\sum_{n=0}^{\infty}\frac{3^n x^n}{n!} = 1 + 3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \frac{(3x)^4}{4!} + ...$$

$$\therefore \quad a_n = \frac{3^n x^n}{n!}$$

$$\therefore \quad a_{n+1} = \frac{3^{n+1}x^{n+1}}{(n+1)!}$$

$$\therefore \quad \frac{a_{n+1}}{a_n} = \frac{3^{n+1}\,x^{n+1}\,n!}{(n+1)!\,3^n\,x^n} = \frac{3x\,n!}{(n+1)\;n!} = \frac{3x}{n+1}$$

$\therefore \quad \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{3x}{n+1}\right| = 3\,|x|\,\frac{1}{n+1}$

$\therefore \quad \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 3\,|x|\,\frac{1}{n+1} = 0 < 1$

$\therefore$ La serie converge en $-\infty < x < \infty$ y el radio es $R = \infty$.

(f) $\sum_{n=1}^{\infty} n^n x^n$.

$$\sum_{n=1}^{\infty} n^n x^n = x + 2^2x^2 + 3^3x^3 + 4^4x^4 + ...$$

$\therefore \quad a_n = n^n x^n$

$\therefore \quad |a_n| = |n^n x^n| = |x|^n\, n^n$

$\therefore \quad \sqrt[n]{|a_n|} = \sqrt[n]{|x|^n\, n^n} = |x|\, n$

$\therefore \quad \lim_{n\to\infty}\sqrt[n]{|a_n|} = |x| \lim_{n\to\infty} n = \begin{cases} 0, & x = 0 \\ \infty, & x \neq 0. \end{cases}$

$\therefore$ La serie sólo converge en $x = 0$ y el radio es $R = 0$.

14. La serie de Taylor generada por $f(x)$ alrededor de x_0 está dada por

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n = f(x_0)+f'(x_0)(x-x_0)+\frac{1}{2}f''(x_0)(x-x_0)^2+\frac{1}{3!}f'''(x_0)(x-x_0)^3+\dots$$

y el correspondiente polinomio de Taylor de orden 2 es la función

$$P_2(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2}f''(x_0)(x-x_0)^2.$$

(a) $f(x) = e^{-2x}, \quad x_0 = 0.$

$$\begin{array}{ll} f(x) = e^{-2x} & \therefore \; f(0) = 1 \\ f'(x) = -2e^{-2x} & \therefore \; f'(0) = -2 \\ f''(x) = 4e^{-2x} & \therefore \; f''(0) = 4 \\ f'''(x) = -8e^{-2x} & \therefore \; f'''(0) = -8 \\ \quad\vdots & \quad\quad\vdots \\ f^{(n)}(x) = (-2)^n e^{-2x} & \therefore \; f^{(n)}(0) = (-2)^n. \end{array}$$

De este modo, la serie de Taylor generada por f alrededor de $x_0 = 0$ es

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(0)\,x^n}{n!} = \sum_{n=0}^{\infty}\frac{(-2)^n x^n}{n!} = \sum_{n=0}^{\infty}\frac{(-2x)^n}{n!} = 1-2x+\frac{(2x)^2}{2!}-\frac{(2x)^3}{3!}+\frac{(2x)^4}{4!}-...$$

El polinomio de Taylor de orden 2 es:

$$P_2(x) = 1 - 2x + 2x^2.$$

(b) $f(x) = e^x, \quad x_0 = -3.$

$$\begin{array}{lll} f(x) = e^x & \therefore & f(-3) = e^{-3} \\ f'(x) = e^x & \therefore & f'(-3) = e^{-3} \\ \vdots & & \vdots \\ f^{(n)}(x) = e^x & \therefore & f^{(n)}(-3) = e^{-3}. \end{array}$$

De este modo, la serie de Taylor generada por f alrededor de $x_0 = -3$ es

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{f^{(n)}(-3)(x-(-3))^n}{n!} &= \sum_{n=0}^{\infty} \frac{e^{-3}(x+3)^n}{n!} = e^{-3} \sum_{n=0}^{\infty} \frac{(x+3)^n}{n!} \\ &= e^{-3}\left[1 + (x+3) + \frac{(x+3)^2}{2!} + \frac{(x+3)^3}{3!} + \ldots\right]. \end{aligned}$$

El polinomio de Taylor de orden 2 es:

$$P_2(x) = e^{-3}\left[1 + (x+3) + \frac{(x+3)^2}{2}\right].$$

(c) $f(x) = x^4 - 2x^3 - 5x + 4, \quad x_0 = 0.$

$$\begin{array}{lll} f(x) = x^4 - 2x^3 - 5x + 4 & \therefore & f(0) = 4 \\ f'(x) = 4x^3 - 6x^2 - 5 & \therefore & f'(0) = -5 \\ f''(x) = 12x^2 - 12x & \therefore & f''(0) = 0 \\ f'''(x) = 24x - 12 & \therefore & f'''(0) = -12 \\ f^{(IV)}(x) = 24 & \therefore & f^{(IV)}(0) = 24 \\ f^{(n)}(x) = 0, \ n \geq 5 & \therefore & f^{(n)}(0) = 0, \ n \geq 5. \end{array}$$

De este modo, la serie de Taylor generada por f alrededor de $x_0 = 0$ es

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)\, x^n}{n!} = (4) + (-5)\, x + (0)\, \frac{x^2}{2} + (-12)\, \frac{x^3}{3!} + (24)\, \frac{x^4}{4!} + 0 = 4 - 5x - 2x^3 + x^4.$$

El polinomio de Taylor de orden 2 es:

$$P_2(x) = 4 - 5x.$$

15. (a) $f(x) = \dfrac{1}{1-x}, \quad x_0 = 0.$

$$\begin{array}{lll} f(x) = \dfrac{1}{1-x} & \therefore & f(0) = 1 \\ f'(x) = \dfrac{1}{(1-x)^2} & \therefore & f'(0) = 1 \\ f''(x) = \dfrac{2}{(1-x)^3} & \therefore & f''(0) = 2. \end{array}$$

$$\begin{aligned} & P_2(x) = f(0) + f'(0)\, x + \tfrac{1}{2} f''(0) x^2 \\ \therefore \quad & P_2(x) = 1 + x + x^2. \end{aligned}$$

(b) $f(x) = 1 + 2x, \qquad x_0 = 0.$

$$\begin{array}{lll} f(x) = 1 + 2x & \therefore & f(0) = 1 \\ f'(x) = 2 & \therefore & f'(0) = 2 \\ f''(x) = 0 & \therefore & f''(0) = 0. \end{array}$$

$$P_2(x) = f(0) + f'(0)x + \tfrac{1}{2}f''(0)x^2$$

$$\therefore \quad P_2(x) = 1 + 2x.$$

Observa que $P_2(x) = f(x)$, ya que f es un polinomio de grado ≤ 2.

(c) $f(x) = \ln(2 + x), \qquad x_0 = -1.$

$$\begin{array}{lll} f(x) = \ln(2+x) & \therefore & f(-1) = 0 \\ f'(x) = \dfrac{1}{2+x} & \therefore & f'(-1) = 1 \\ f''(x) = -\dfrac{1}{(2+x)^2} & \therefore & f''(-1) = -1. \end{array}$$

$$P_2(x) = f(-1) + f'(-1)(x+1) + \frac{1}{2}f''(-1)(x+1)^2$$

$$\therefore \quad P_2(x) = (x+1) - \frac{1}{2}(x+1)^2.$$

(d) $f(x) = 2 + \int_3^{3x} e^{9-t^2}\, dt, \qquad x_0 = 1.$

$$\begin{array}{lll} f(x) = 2 + \int_3^{3x} e^{9-t^2}\, dt & \therefore & f(1) = 2 + \int_3^{3} e^{9-t^2}\, dt = 2. \\ f'(x) = 3e^{9-(3x)^2} = 3e^{9-9x^2} & \therefore & f'(1) = 3 \\ f''(x) = -54xe^{9-9x^2} & \therefore & f''(1) = -54. \end{array}$$

$$P_2(x) = f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2$$

$$\therefore \quad P_2(x) = 2 + 3(x-1) - 27(x-1)^2.$$

16. (a) $f(x) = \sqrt{4+x}, \quad x_0 = 0.$

$$\begin{array}{lll} f(x) = \sqrt{4+x} & \therefore & f(0) = 2 \\ f'(x) = \dfrac{1}{2\sqrt{4+x}} & \therefore & f'(0) = \dfrac{1}{2(2)} = \dfrac{1}{4} \\ f''(x) = -\dfrac{1}{4(4+x)^{3/2}} & \therefore & f''(0) = -\dfrac{1}{4(4)^{3/2}} = -\dfrac{1}{4(2)^3} = -\dfrac{1}{32}. \end{array}$$

$$\therefore \quad P_2(x) = 2 + \frac{1}{4}x - \frac{1}{64}x^2$$

$$\therefore \quad \sqrt{4+x} \cong 2 + \frac{1}{4}x - \frac{1}{64}x^2.$$

(b) $$\sqrt{4.2} = \sqrt{4 + (0.2)} \cong 2 + \frac{0.2}{4} - \frac{(0.2)^2}{64} = \underbrace{2.049375}_{\text{compara con } 2.0493901}.$$

(c) $$\sqrt{4+x} \cong 2 + \frac{1}{4}x - \frac{1}{64}x^2, \text{ para } x \text{ cercano a } 0.$$

$$\therefore \quad \sqrt{4+2x^2} \cong 2 + \frac{1}{4}(2x^2) - \frac{1}{64}(2x^2)^2 = 2 + \frac{1}{2}x^2 - \frac{1}{16}x^4.$$

17. (a) $f(x) = e^x, \qquad x_0 = 0.$

$$\begin{array}{lll} f(x) = e^x & \therefore & f(0) = 1 \\ f'(x) = e^x & \therefore & f'(0) = 1 \\ f''(x) = e^x & \therefore & f''(0) = 1. \end{array}$$

$$\therefore \quad P_2(x) = 1 + x + \frac{x^2}{2}$$

$$\therefore \quad e^x \cong 1 + x + \frac{x^2}{2}.$$

(b) $$e^{0.15} \cong 1 + (0.15) + \frac{(0.15)^2}{2} = \underbrace{1.16125}_{\text{compara con } 1.16183424}.$$

(c) $$e^x \cong 1 + x + \frac{x^2}{2}, \text{ para } x \text{ cercano a } 0.$$

$$\therefore \quad e^{-3x} \cong 1 + (-3x) + \frac{(-3x)^2}{2} = 1 - 3x + \frac{9x^2}{2}.$$

18. La serie de Taylor generada por $f(x) = e^x$ alrededor de $x_0 = 0$ es $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. Por lo tanto,

(a) $$1 + \ln 2 + \frac{(\ln 2)^2}{2!} + + \frac{(\ln 2)^3}{3!} + \frac{(\ln 2)^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{(\ln 2)^n}{n!} = e^{\ln 2} = 2.$$

(b) $$\sum_{n=0}^{\infty} \frac{\lambda^n e^{-\lambda}}{n!} = e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} = e^{-\lambda} e^{\lambda} = 1.$$

19. El polinomio de Taylor de orden 2 generado por $f(x, y)$ alrededor de (x_0, y_0) es

$$\begin{aligned} P_2(x, y) &= f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\ &\quad + \frac{1}{2}\left[f_{xx}(x_0, y_0)(x - x_0)^2 + 2f_{xy}(x_0, y_0)(x - x_0)(y - y_0) + f_{yy}(x_0, y_0)(y - y_0)^2\right]. \end{aligned}$$

(a) $f(x, y) = e^x \ln(1 + y), \qquad (x_0, y_0) = (0, 0).$

$$\begin{array}{lll} f(x, y) = e^x \ln(1 + y) & \therefore & f(0, 0) = 0 \\ f_x(x, y) = e^x \ln(1 + y) & \therefore & f_x(0, 0) = 0 \\ f_y(x, y) = \dfrac{e^x}{1 + y} & \therefore & f_y(0, 0) = 1 \\ f_{xx}(x, y) = e^x \ln(1 + y) & \therefore & f_{xx}(0, 0) = 0 \\ f_{xy}(x, y) = \dfrac{e^x}{1 + y} & \therefore & f_{xy}(0, 0) = 1 \\ f_{yy}(x, y) = -\dfrac{e^x}{(1 + y)^2} & \therefore & f_{yy}(0, 0) = -1. \end{array}$$

$$\begin{aligned} \therefore \quad P_2(x, y) &= 0 + (0)(x - 0) + (1)(y - 0) \\ &\quad + \frac{1}{2}\left[(0)(x - 0)^2 + 2(1)(x - 0)(y - 0) + (-1)(y - 0)^2\right] \end{aligned}$$

$$\therefore \quad P_2(x, y) = y + xy - \frac{1}{2}y^2.$$

(b) $f(x,y) = 3x^4 + 14x^3y + y^3x^2, \qquad (x_0, y_0) = (2, -1)$.

$$\begin{array}{lll} f(x,y) = 3x^4 + 14x^3y + y^3x^2 & \therefore & f(2,-1) = -68 \\ f_x(x,y) = 12x^3 + 42x^2y + 2y^3x & \therefore & f_x(2,-1) = -76 \\ f_y(x,y) = 14x^3 + 3y^2x^2 & \therefore & f_y(2,-1) = 124 \\ f_{xx}(x,y) = 36x^2 + 84xy + 2y^3 & \therefore & f_{xx}(2,-1) = -26 \\ f_{xy}(x,y) = 42x^2 + 6y^2x & \therefore & f_{xy}(2,-1) = 180 \\ f_{yy}(x,y) = 6yx^2 & \therefore & f_{yy}(2,-1) = -24. \end{array}$$

$$\therefore \quad P_2(x,y) = -68 + (-76)(x-2) + (124)(y+1) + \frac{1}{2}\left[(-26)(x-2)^2 + 2(180)(x-2)(y+1) + (-24)(y+1)^2\right]$$

$$\therefore \quad P_2(x,y) = -68 - 76(x-2) + 124(y+1) - 13(x-2)^2 + 180(x-2)(y+1) - 12(y+1)^2.$$

(c) $f(x,y) = \int_x^{2y} e^{t^2}\, dt, \qquad (x_0, y_0) = (2, 1)$.

$$\begin{array}{lll} f(x,y) = \int_x^{2y} e^{t^2}\, dt & \therefore & f(2,1) = \int_2^{2} e^{t^2}\, dt = 0 \\ f_x(x,y) = -e^{x^2} & \therefore & f_x(2,1) = -e^4 \\ f_y(x,y) = 2e^{4y^2} & \therefore & f_y(2,1) = 2e^4 \\ f_{xx}(x,y) = -2xe^{x^2} & \therefore & f_{xx}(2,1) = -4e^4 \\ f_{xy}(x,y) = 0 & \therefore & f_{xy}(2,1) = 0 \\ f_{yy}(x,y) = 16ye^{4y^2} & \therefore & f_{yy}(2,1) = 16e^4. \end{array}$$

$$\therefore \quad P_2(x,y) = 0 + \left(-e^4\right)(x-2) + \left(2e^4\right)(y-1) + \frac{1}{2}\left[\left(-4e^4\right)(x-2)^2 + (0)(x-2)(y-1) + \left(16e^4\right)(y-1)^2\right]$$

$$\therefore \quad P_2(x,y) = e^4\left[-(x-2) + 2(y-1) - 2(x-2)^2 + 8(y-1)^2\right].$$

20. El punto crítico (x_0, y_0) de $f(x,y) = xy - x^2 - y^2 + 3y$ satisface

$$\begin{array}{lllll} f_x(x,y) & = & y - 2x & = & 0 \\ f_y(x,y) & = & x - 2y + 3 & = & 0, \end{array}$$

de donde $(x_0, y_0) = (1, 2)$.

Se tiene:

$$\begin{array}{lll} f(x,y) = xy - x^2 - y^2 + 3y & \therefore & f(1,2) = 3 \\ f_x(x,y) = y - 2x & \therefore & f_x(1,2) = 0 \\ f_y(x,y) = x - 2y + 3 & \therefore & f_y(1,2) = 0 \\ f_{xx}(x,y) = -2 & \therefore & f_{xx}(1,2) = -2 \\ f_{xy}(x,y) = 1 & \therefore & f_{xy}(1,2) = 1 \\ f_{yy}(x,y) = -2 & \therefore & f_{yy}(1,2) = -2. \end{array}$$

$$\therefore \quad P_2(x,y) = 3 + (0)(x-1) + (0)(y-2) + \frac{1}{2}\left[(-2)(x-1)^2 + 2(1)(x-1)(y-2) + (-2)(y-2)^2\right]$$

$$\therefore \quad P_2(x,y) = 3 - (x-1)^2 + (x-1)(y-2) - (y-2)^2.$$